Number-phase uncertainty relations: Verification by balanced homodyne measurement

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It is shown that fundamental uncertainty relations between photon number and canonical phase of a single-mode optical field can be verified by means of a balanced homodyne measurement. All the relevant quantities can be sampled directly from the measured phase-dependent quadrature distribution. [S1050-2947(98)04003-7]

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Although the problem of number-phase uncertainty has been studied widely, there has been no direct experimental verification of fundamental uncertainty relations (URs). What can be the best way of doing that? A powerful and perhaps ultimate method for measuring the quantum statistics of traveling optical fields has been balanced homodyne detection. The quantity that is directly measured is the probability distribution \( p(x, \theta) \) of the phase-dependent quadrature \( \hat{x}(\theta) = 2^{-1/2} (\hat{a}e^{-i\theta} + \hat{a}^\dagger e^{i\theta}) \), where \( \hat{a} \) (\( \hat{a}^\dagger \)) is the bosonic annihilation (creation) operator of the (single-mode) signal field and \( \theta \) corresponds to the local-oscillator phase. It has been shown that \( p(x, \theta) \) for all phases \( \theta \) in a \( \pi \) interval contains all knowable information about the quantum state of the signal field and can be used to reconstruct the Wigner function by applying the standard filtered back projection algorithm in the numerical calculation of the inverse Radon transform to be performed [1].

Since the Wigner function is a full description of the quantum state, it can be used to calculate other important features of the field such as the photon-number statistics and phase statistics and their associated URs [2]. The inverse Radon transform requires a threefold integration of the measured data and the calculation of the density matrix in the Fock basis can then be accomplished with two integrals. One summation eventually yields the photon-number moments, and one sum and one integral must be performed to obtain the (canonical) phase moments. Hence six- and sevenfold transformations of the recorded data are required for UR verification at least. Of course, a large amount of data manipulation accumulates various errors and the physical nature of the uncertainties becomes less transparent.

Recent progress has offered possibilities of determining the photon-number statistics in a more direct way avoiding the detour via the Wigner function. It has been shown that both the density-matrix elements \( \rho_{nn} \) in the Fock basis [3] and the moments and correlations \( \langle \hat{a}^{\dagger k} \hat{a}^{k'} \rangle \) [4] can be sampled directly from the recorded data according to a two-fold integral transform

\[
\mathcal{A} = \int_{2\pi} d\theta \int_{-\infty}^{\infty} dx K_{\alpha}(x, \theta) p(x, \theta).
\]

Here \( \mathcal{A} \) is the quantity that is to be determined and \( K_{\alpha}(x, \theta) \) is the corresponding integral kernel (sampling function). In particular, \( \mathcal{A} \) can be identified with \( p_n = \rho_{nn} \) or \( \langle \hat{n}^k \rangle \).

In contrast to the photon number, the phase has remained perhaps ultimate method for measuring the quantum statistics of traveling optical fields has been balanced homodyne detection. The quantity that is directly measured is the probability distribution \( p(x, \theta) \) of the phase-dependent quadrature \( \hat{x}(\theta) = 2^{-1/2} (\hat{a}e^{-i\theta} + \hat{a}^\dagger e^{i\theta}) \), where \( \hat{a} \) (\( \hat{a}^\dagger \)) is the bosonic annihilation (creation) operator of the (single-mode) signal field and \( \theta \) corresponds to the local-oscillator phase. It has been shown that \( p(x, \theta) \) for all phases \( \theta \) in a \( \pi \) interval contains all knowable information about the quantum state of the signal field and can be used to reconstruct the Wigner function by applying the standard filtered back projection algorithm in the numerical calculation of the inverse Radon transform to be performed [1].

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In particular, for the canonical phase the relation \( \psi_k = \langle \hat{E}^k \rangle \) is valid, where the exponential phase operator \( \hat{E} \) is given by \( \hat{E} = (\hat{n} + 1)^{-1/2} \hat{a} \) (for the canonical phase and the exponential phase operator; see, e.g., [7]). In order to define a mean phase \( \varphi \) that is independent of the chosen phase window, the first-order exponential phase moment has been introduced in the definition \( \varphi = \arg \psi_1 = \arg \langle \hat{E} \rangle \) and it has been used to define phase uncertainty measures such as

\[
\psi_k = \int_{2\pi} d\varphi e^{ik\varphi} p(\varphi).
\]
\[ \Delta \varphi = \arccos|\Psi_1| = \arccos\langle \hat{E} \rangle \]  
\[ \sigma_{BP} = \sin \Delta \varphi, \sigma_H = \tan \Delta \varphi. \]  

It can then be proved that the UR

\[ \Delta n \tan \Delta \varphi \geq \frac{1}{4} \]  

is valid [8], which is equivalent to the Holevo UR [10]

\[ (\Delta n)^2 \sigma_H \geq \frac{1}{4} \]  

\[ (\Delta n)^2 = \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2 \]. Although these URs are exact, they are, in a sense, weak. This means that they also allow for such values of \( \Delta n \) and \( \Delta \varphi \) for which no state exists (for more specific URs and the corresponding minimizing states, see [11,12]). Further, URs that are based on the Susskind-Glogower trigonometric operators \( \hat{C} = \frac{1}{2}(\hat{E} + \hat{E}^\dagger) \) and \( \hat{S} = (1/2i)(\hat{E} - \hat{E}^\dagger) \) have also been studied [7,12,13],

\[ \Delta n \Delta C \geq \frac{1}{2} \langle \hat{S} \rangle, \quad \Delta n \Delta S \geq \frac{1}{2} \langle \hat{C} \rangle, \]

\[ \Delta \Delta C \geq \frac{1}{4} \langle \hat{S} \rangle, \quad \Delta \Delta S \geq \frac{1}{4} \langle \hat{C} \rangle. \]  

From the definitions of \( \Delta \varphi \), \( \Delta C \), and \( \Delta S \) we see that to measure them it is sufficient to measure the exponential phase moments \( \Psi_1 = \langle \hat{E} \rangle \) and \( \Psi_2 = \langle \hat{E}^2 \rangle \) and the density-matrix element \( \varrho_{00} = \langle \langle 0 | 0 \rangle \rangle. \) The determination of the photon-number uncertainty requires measurement of \( \langle \hat{n} \rangle \) and \( \langle \hat{n}^2 \rangle \).

It is well known that \( \varrho_{00}, \langle \hat{n} \rangle \), and \( \langle \hat{n}^2 \rangle \) can be sampled directly from the data recorded in balanced homodyning in applying Eq. (1). The kernel \( K_0(x, \vartheta) \) for \( \varrho_{00} \) can be taken from [3], \( K_0(x, \vartheta) = \pi^{-1} \Phi(1, \frac{1}{2}, -x^2) \), where \( \Phi(a,b,z) \) is the confluent hypergeometric function. The kernels \( K_n(x, \vartheta) \) and \( K_{2n}(x, \vartheta) \) for \( \langle \hat{n} \rangle \) and \( \langle \hat{n}^2 \rangle \), respectively, can simply be obtained from the sampling formula for the normally ordered moments and correlations of bosonic operators [4]

\[ \langle \hat{a}^n \hat{a}^m \rangle = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} d\vartheta e^{-i(n-m)\vartheta} \frac{H_{n+m}(x)}{2\pi \sqrt{2\pi}} \left( \begin{array}{c} n+m \cr m \end{array} \right) p(x, \vartheta), \]

from which we find that

\[ K_n(x, \vartheta) = \frac{1}{2\pi} \left( x^2 - \frac{1}{2} \right), \]  

\[ K_{2n}(x, \vartheta) = \frac{1}{2\pi} \left( \frac{2}{3} x^4 - x^2 \right). \]  

Note that for large \( |x| \) the leading terms in these expressions determine the kernels for the energy moments in the classical limit.

Let us now turn to the problem of direct sampling of the exponential phase moments of the canonical phase. Provided that the corresponding integral kernels exist, their asymptotic behavior for large \( |x| \) can be obtained from considering the classical limit. Since in classical physics the phase probability distribution \( p(\varphi) \) can be obtained from the phase-space probability distribution \( W(r, \varphi) \) according to \( p(\varphi) = \int_0^\infty r \, dr \, W(r, \varphi) \), the exponential phase moments \( \Psi_k \) can be given by

\[ \Psi_k = \int_{2\pi} d\varphi \int_0^\infty r \, dr \, W(r, \varphi) e^{ik\varphi}. \]

Further, the quadrature probability \( p(x, \vartheta) \) is given by the Radon transform

\[ p(x, \vartheta) = \int_{2\pi} d\vartheta \int_0^\infty r \, dr \, W(r, \varphi) \delta(x - r \cos(\vartheta - \varphi)). \]

Let us now assume that \( \Psi_k \) can be related to \( p(x, \vartheta) \) according to Eq. (1) \([ A \rightarrow \Psi_k \) and \( K_{2k}(x, \vartheta) \rightarrow K_{2k}(x, \vartheta) \]). Substituting in this equation for \( p(x, \vartheta) \) the result of Eq. (15) and comparing with Eq. (14), we observe that \( K_k(x, \vartheta) \) can be written as

\[ K_k(x, \vartheta) = e^{ik\vartheta} K_k(x), \]

where \( K_k(x) \) must satisfy the integral equation

\[ \int_{2\pi} d\varphi e^{ik\varphi} K_k(r \cos \varphi) = 1 \]

for any \( r > 0 \). From Eq. (17) we can see that \( K_k(x) \) is not uniquely defined. First, any function of parity \( (-1)^{k+1} \) can be added to \( K_k(x) \) without changing the integral. Second, any polynomial of a degree less than \( k \) can also be added to \( K_k(x) \). As can be verified by direct substitution, Eq. (17) is solved using the functions

\[ K_{2m+1}(x) = \frac{1}{2} (-1)^m (2m + 1) \text{sgn}(x) \]

and

\[ K_{2m}(x) = \pi^{-1} (-1)^{m+1} m \ln|x| \]

\((m=0,1,2, \ldots)\). It is worth noting that since Eq. (15) is also valid when \( W(r, \varphi) \) is the quantum-mechanical Wigner function, using in homodyne detection the kernels (18) and (19) over the whole \( x \) axis would yield the exponential phase moments of the phase quasiprobability distribution defined by the radially integrated Wigner function.
With regard to the canonical phase, the kernels (18) and (19) are of course valid only for large $|x|$. To obtain them for arbitrary $|x|$, we recall that in quantum physics $\Psi_k$ can be given by

$$\Psi_k = \langle \hat{E}^k \rangle = \sum_{n=0}^{\infty} \mathcal{Q}_{n+k,n}$$

(20)

in place of Eq. (14). Expressing $p(x, \vartheta)$ in terms of the density-matrix elements $\mathcal{Q}_{nn'}$ as

$$p(x, \vartheta) = \sum_{n,n'=0}^{\infty} e^{i(n'-n)\vartheta} \psi_n(x) \psi_{n'}(x) \mathcal{Q}_{nn'}$$

(21)

and assuming that Eq. (1) applies to $\Psi_k$, we again find that $K_k(x, \vartheta) = e^{ik\vartheta}K_k(x)$ [Eq. (16)] but in place of Eq. (17) $K_k(x)$ must now satisfy the integral equation

$$2\pi \int_{-\infty}^{\infty} dx \ K_k(x) \psi_n(x) \psi_{n+k}(x) = 1$$

(22)

for every $n=0, 1, 2, \ldots$. Here $\psi_n(x) = (2^n n! \sqrt{\pi})^{-1/2} \exp(-x^2/2) H_n(x)$ are the energy eigenfunctions of a harmonic oscillator, with $H_n(x)$ being the Hermite polynomials. From Eq. (22) and the properties of the Hermite polynomials [14] the same arbitrariness in the determination of $K_k(x)$ as in the classical limit is found. To derive an explicit expression, we use the expansion

$$\hat{E}^k = \sum_{n=0}^{\infty} \frac{\hat{a}^n \exp(-\hat{a}^2) \hat{a}^{n+k}}{\sqrt{n!(n+k)!}}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{1}{\sqrt{n!(n+k)!}} (-1)^m \hat{a}^{n+m} \hat{a}^{n+m+k}$$

(23)

(: introduces normal order) and apply Eq. (11) in order to represent $\langle \hat{E}^k \rangle$ in the form of Eq. (1). Provided all the correlations $\langle \hat{a}^n \hat{a}^{n+k} \rangle$ exist, we derive

$$K_k(x) = (2\pi)^{-1} \sum_{l=0}^{\infty} C_l^{(k)} H_{2l+k}(x),$$

(24)

where

$$C_l^{(k)} = \frac{(l+k)!}{2^{l+(k/2)}(2l+k)!} \sum_{n=0}^{l} \frac{(-1)^{l-n}}{\sqrt{(n+1)\cdots(n+k)}}.$$

(25)

Using in Eq. (25) the relation

$$\frac{1}{\sqrt{n}} = \frac{1}{\sqrt{n}} \int_{-\infty}^{+\infty} dt e^{-nt^2},$$

(26)

we can perform the summation over the Hermite polynomials in Eq. (24), on using standard formulas [14]. For $k = 1$ we arrive at

$$K_1(x) = \pi^{-3/2} x \int_{0}^{\infty} \frac{dt}{\sqrt{\cosh^2 t}} \Phi(2, \frac{1}{2}, -x^2 \tanh t)$$

(27)

![FIG. 1. The space-dependent parts (a) $K_1(x)$ and (b) $K_2(x)$ of the kernels for sampling the first and second exponential-phase moments are shown (full lines) and compared with the classical results (dashed lines) as given in Eq. (18) and (up to an irrelevant constant) in Eq. (19), respectively. (c) The kernel $K_{C2}(x, \vartheta)$ for sampling $\langle \hat{C}^2 \rangle$ is shown for $\vartheta = 0$ (1), $\vartheta = \pi/4$ (2), and $\vartheta = \pi/2$ (3).]

(also see [15]). For $k = 2$ relation (26) must be used both for $1/\sqrt{n}$ and for $1/\sqrt{n+1}$. Introducing polar coordinates in the resulting double integral and integrating over the angular variable, we obtain

$$K_2(x) = \frac{1}{2\pi} \int_{0}^{\infty} dt J_0(t) \left[ e^{-2t} \sinh t - \frac{1}{\cosh^2 t \sinh t} \times \Phi(2, \frac{1}{2}, -x^2 \tanh t) \right]$$

(28)

$I_0(t)$ is the modified Bessel function. Integral representations of the higher-order kernels can be derived accordingly. It can be proven that the kernels exist and satisfy condition (22), i.e., we have found solutions even when the assumption of finite moments fails and Eq. (11) cannot be used. Finally,
it can be proved that in the limit of large $|x|$ the kernels correspond, within the arbitrariness mentioned, to the classical results (18) and (19).

The kernels $K_1(x)$ and $K_2(x)$ can be evaluated numerically using standard routines. They are plotted in Figs. 1(a) and 1(b). As expected, they rapidly approach the classical limits given in Eqs. (18) and (19) as $|x|$ increases. Since they differ from the classical limits only in a small region (of a few ‘‘vacuum-fluctuation widths’’) around zero, in practice their evaluation requires the application of Eqs. (27) and (28) [or Eq. (25)] only for small values of $|x|$, whereas for greater values the expressions given in Eqs. (18) and (19) can be used. Note that for small values of $|x|$ power series expansion of the confluent hypergeometric function in Eqs. (27) and (28) can be used.

To verify URs connected with $\Delta \varphi$ [e.g., relations (5) and (6)], the first moment of $\dot{E}$ must be measured, which can be accomplished with the kernel $K_1(x, \vartheta)$. With regard to URs of the type given in Eqs. (7) and (8), one also needs the second moment of $\dot{E}$ and the vacuum probability $\varrho_{00}$. From the above, the kernels for sampling $\langle \dot{E}^2 \rangle$ and $\langle \dot{S}^2 \rangle$ read

$$K_\pm(x, \vartheta) = \frac{1}{4\pi}[1 - \Phi(1, x^2)] \pm i \cos(2 \vartheta) K_2(x)$$

(29)

[see Fig. 1(c)], where $K_+(x, \vartheta) = K_{\dot{E}^2}(x, \vartheta)$ and $K_-(x, \vartheta) = K_{\dot{S}^2}(x, \vartheta)$.

Let us comment on the necessary introduction of a discrete grid of $\vartheta$ and $x$ values in an experiment. The numbers of $\vartheta$ and $x$ values of course depend, for chosen accuracy, on the actual state and the order of the phase moment considered. In particular, from Eq. (16) it is easily seen that with increasing $k$ the number of phases must also be increased. Compared to the sampling functions for the determination of the density matrix in the Fock basis, the sampling functions for the determination of the exponential phase moments are slowly varying with $x$, so that the number of $x$ values can be reduced drastically. The problem of systematic error associated with the discretization can be treated in a way similar to that described in [6,16]. In particular, an iterative strategy can be adopted, in which the systematic error is reduced step by step until it is below the statistical one.

To conclude, we have presented a method for verification of number-phase URs. It is based on the possibility of direct sampling of exponential phase moments of the canonical phase of a single-mode quantum state $\Psi_k = \langle \hat{E}^k \rangle$ from the data recorded in balanced homodyning. We have shown that the corresponding kernels $K_k(x, \vartheta)$ are state independent and well behaved. With increasing $|x|$ they rapidly approach the classical limits, so that the method applies to both quantum and classical systems in a unified way. Since the method does not only apply to the determination of low-order moments, it may also be used for reconstructing the canonical phase distribution $p(\varphi)$ as a whole. However, a direct (state-independent) sampling of $p(\varphi)$ according to Eq. (1) seems to be impossible. If there would exist a corresponding kernel, its Fourier components with respect to $\vartheta$ would be equal to the kernels for determining $\Psi_k$. Since for chosen $x$ the absolute values of these kernels increase with $k$ [cf. Eqs. (18) and (19)], they cannot be treated as Fourier coefficients of a well-behaved function of $\vartheta$. Of course, this does not exclude an indirect reconstruction of $p(\varphi)$. Measuring a limited number of $\Psi_k$, one can use, e.g., the maximum entropy principle [17] to obtain a $p(\varphi)$ that best fits the measured values without introducing any arbitrary bias. This also offers the possibility of verification of URs that are not based on exponential phase moments [18,19].

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