Density-matrix reconstruction by unbalanced homodyning

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Recently a method for measuring quasi-distributions of single-mode optical fields in unbalanced homodyne detection has been proposed [S. Wallentowitz and W. Vogel, Phys. Rev. A 53, 4528 (1996); K. Banaszek and K. Wódkiewicz, Phys. Rev. Lett. 76, 4344 (1996)]. We show that the scheme can be used for direct sampling of the signal-mode density matrix in the Fock basis. In this case it is sufficient to vary only the phase of the local oscillator, keeping its amplitude constant. [S1050-2947(97)04202-9]

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In two recent papers [1,2], a method for measuring the quantum state of optical single-mode fields has been suggested which is based on unbalanced homodyne detection. When a signal mode and a local oscillator (LO) are mixed by a beam splitter with high transmittance, \( |T| \rightarrow 1 \), and low reflectance, \(|R| \rightarrow 0\), then the transmitted signal mode may be regarded as the signal mode shifted by the displacement operator \( \hat{D}(\alpha) = \exp(\alpha \hat{a}^{\dagger} - \alpha^{*} \hat{a}) \), where \( \hat{a} \) (\( \hat{a}^{\dagger} \)) is the photon destruction (creation) operator associated with the signal mode and \( \alpha = |\alpha| e^{i\varphi} \) is related to the LO amplitude \( \beta \) as \( \alpha = -\beta R/T \). Hence measuring the photon statistics of the transmitted signal is equivalent to measuring the displaced Fock-state probability distribution of the signal,

\[
p_{n}(\alpha) = \langle n, \alpha | \hat{q} | n, \alpha \rangle,
\]

where \( |n, \alpha \rangle = \hat{D}(\alpha) |n \rangle \) are the displaced Fock states (\( \hat{q} \) is the signal-mode density operator; \( \hat{a}^{\dagger} |n \rangle = n |n \rangle \)). The distribution \( p_{n}(\alpha) \) can then be used to determine the values of the \( s \)-parametrized quasi-distributions of the signal mode at the phase-space point \( \alpha \) in a straightforward way. Varying the values of the real and imaginary parts of \( \alpha \) from measurement to measurement, the signal-mode quantum state can be reconstructed in terms of quasi-distributions on a sufficiently dense grid of points in the phase space \([1,2]\). In what follows we present a method for direct sampling of the density matrix in the Fock basis. We show that it is not necessary to trace the whole phase space, but it is sufficient to keep the absolute value of the LO amplitude constant and change only its phase.

Expanding in Eq. (1) the density operator \( \hat{q} \) in the Fock basis, we can relate the measured photon statistics \( p_{n}(\alpha) \) to the density-matrix elements of the signal mode as

\[
p_{n}(\alpha) = \sum_{k,m=0}^{\infty} \langle n, \alpha | k \rangle \langle k | \hat{q} | m \rangle \langle m | n, \alpha \rangle.
\]

For any physical state the density-matrix elements \( \langle k | \hat{q} | m \rangle \) must eventually decrease indefinitely with increasing \( k \) and \( m \). It therefore follows that \( p_{n}(\alpha) \) can always be approximated to any desired degree of accuracy by setting \( \langle k | \hat{q} | m \rangle \) to zero for \( k, m > n_{0} \), if \( n_{0} \) is suitably large. This allows us to truncate the sum in Eq. (2), i.e.,

\[
p_{n}(\alpha) = \sum_{k,m=0}^{n_{0}} \langle n, \alpha | k \rangle \langle k | \hat{q} | m \rangle \langle m | n, \alpha \rangle.
\]

Expressing the displaced Fock states \( |n, \alpha \rangle \) in the ordinary Fock basis [3],

\[
\langle m | n, \alpha \rangle = \exp \left( -\frac{1}{2} |\alpha|^{2} \right) \sqrt{m! n!} \times \prod_{l=0}^{n-1} \frac{(-1)^{n-l} |\alpha|^{m+n-2l} e^{i(m-n)\varphi}}{l! (m-l)! (n-l)!},
\]

where \( \{m,n\} = \min(m,n) \), we can rewrite \( p_{n}(\alpha) \) as

\[
p_{n}(\alpha) = \sum_{k,m=0}^{n_{0}} \sqrt{k! m!} \langle k | \hat{q} | m \rangle \times \prod_{l=0}^{n-1} \frac{(-1)^{n-l} |\alpha|^{m+k+2(n-j-l)} e^{i(m-k)\varphi}}{j! (n-j)! (k-j)! (m-l)! (n-l)!}. \]

For chosen value of \( |\alpha| \) we now regard \( p_{n}(\alpha) = p_{n}(|\alpha| e^{i\varphi}) \) as a function of \( \varphi \) and calculate the coefficients of the Fourier-series expansion,

\[
p_{n}(\alpha) = \frac{1}{2\pi} \int_{0}^{2\pi} d\varphi \ p_{n}(\alpha) e^{i\varphi} \]

\((s=0,1,2,\ldots)\). Combining Eqs. (5) and (6), we find that

\[
p_{n}(\alpha) = \sum_{m=0}^{n_{0}-s} C_{n,m}(|\alpha|) \langle m + s | \hat{q} | m \rangle,
\]

where

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We see that $p_n^{(s)}(|\alpha|)$ is related to the density-matrix elements whose row and column indices differ by $s$.

Let us assume that the photon-number distribution $p_n(\alpha)$ is measured for $n=0,1, \ldots, N$, with $N \gg n_0$. Hence Eq. (7) represents for each $s$ a system of $(N+1)$ linear equations between $(N+1)$ measured quantities $p_n^{(s)}$ and $(n_0+1-s)$ unknown density-matrix elements. To obtain the density-matrix elements from the measured probabilities, we can invert this overdetermined system using the method of least squares. The reconstructed density matrix is then derived from the measured probabilities as

$$
\langle m+s|\hat{\rho}^{\mathrm{rec}}|m\rangle = \sum_{n=0}^{N} F_{m,n}^{(s)}(|\alpha|) p_n^{(s)}(|\alpha|),
$$

(9)

where the $F$ matrices are given as $F=(G^TG)^{-1}G^T$ (see, e.g., Ref. [4]). Here $G$ and $F$ are finite matrices, and the symbol $T$ introduces matrix transposition. As can be seen, the $F$ matrices satisfy the conditions that

$$
\sum_{n=0}^{N} F_{m,n}^{(s)}(|\alpha|) G_{n,m}^{(s)}(|\alpha|) = \delta_{m,m'},
$$

(10)

for $m, m'=0,1, \ldots, n_0-s$, which means that from the exact probabilities satisfying Eq. (8) the correct density matrix is obtained, $\hat{\rho}^{\mathrm{rec}} = \hat{\rho}$. The least-squares method ensures that when the measured values of $p_n^{(s)}$ are inaccurate, then the quantities $p_n^{(s)}$ calculated from $\hat{\rho}^{\mathrm{rec}}$ as

$$
p_n^{(s)} = \sum_{n=0}^{N} G_{n,m}^{(s)}(|\alpha|) \langle m+s|\hat{\rho}^{\mathrm{rec}}|m\rangle
$$

best fit the measured quantities such that $\sum_{n=0}^{N} (p_n^{(s)}-p_n^{(s)})^2$ is minimized. Combining Eqs. (9) and (6), we find that

$$
\langle m+s|\hat{\rho}^{\mathrm{rec}}|m\rangle = \frac{1}{2\pi} \sum_{n=0}^{N} \int d\varphi F_{m,n}^{(s)}(|\alpha|) e^{i\varphi} p_n(\alpha),
$$

(11)

which may be regarded as the basic equations for direct sampling of the density matrix from the measured distributions $p_n(\alpha)$. In a sense, it is the analog of the sampling formulas in balanced four-port and eight-port (double) homodyne detectors, respectively, as given in Refs. [5–7]. In particular, we see that the determination of the quantum states only requires the value of $\varphi$ to be varied.

An extension of the above given results to nonperfect detection is straightforward. When the detection efficiency $\eta$ is less than unity, then the measured photocount distribution $P_k(\alpha)$ can be related to the photon-number distribution $p_n(\alpha)$ as

$$
P_k(\alpha) = \sum_{n=0}^{\infty} M_{k,n}(\eta) p_n(\alpha),
$$

(12)

for $k \leq n$ and $M_{k,n}(\eta) = 0$ for $k>n$. Equation (7) is then modified as

$$
P_n^{(s)}(|\alpha|) = \sum_{m=0}^{n_0-s} G_{m,n}^{(s)}(|\alpha|,\eta) \langle m+s|\hat{\rho}|m\rangle,
$$

(13)

where $P_n^{(s)}(|\alpha|)$ is again defined according to Eq. (6), but now with $P_n(\alpha)$ in place of $p_n(\alpha)$, and $G_{n,m}^{(s)}(|\alpha|,\eta)$ reads as

$$
G_{n,m}^{(s)}(|\alpha|,\eta) = \sum_{k=0}^{\infty} M_{k,n}(\eta) G_{k,m}^{(s)}(|\alpha|),
$$

(14)

with $G_{k,m}^{(s)}(|\alpha|)$ from Eq. (8). The matrices $G_{n,m}^{(s)}(|\alpha|,\eta)$ can then be inverted in the way described above to obtain matrices $F_{n,m}^{(s)}(|\alpha|,\eta)$ that can be used, according to Eq. (11), to reconstruct the density matrix.

Let us comment on the experimental conditions. To reconstruct the density matrix with sufficiently good precision from a reasonably large set of measurement events, a highly efficient detector that is able to distinguish between single photons is needed. Unfortunately, such detectors are not available at present. To overcome the difficulty, in Ref. [1] it is suggested to defocus the measured beam over an array of highly efficient avalanche photodiodes (which are saturated by a single photon), so that the probability for illuminating one photodiode by more than one photon is negligible. The method is similar to the proposed photon-chopping scheme [9]. Here the mode to be measured is used as an input of a $(2N_2)$ port. The remaining $(N_2-1)$ input ports are empty. Following Ref. [9], the photon-number distribution $P_n(\alpha)$ can be obtained from the probability distribution $P_{\alpha}(\alpha)$ of recording $n$ coincident events of $N_\alpha$ avalanche photodiodes (with efficiencies $\eta$) in the output channels as

$$
P_{\alpha}(\alpha) = \sum_{m=0}^{\infty} C_{m,n}(N_\alpha) P_m(\alpha),
$$

(15)

$$
C_{m,n}(N_\alpha) = \frac{1}{(N_\alpha)^m} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^m
$$

(16)

for $n \leq m$, and $C_{m,n}(N_\alpha) = 0$ for $n > m$. Hence replacing $G_{n,m}^{(s)}(|\alpha|,\eta)$ with

$$
G_{n,m}^{(s)}(|\alpha|,\eta,N_\alpha) = \sum_{k=0}^{\infty} C_{k,n}(N_\alpha) G_{k,m}^{(s)}(|\alpha|,\eta),
$$

(17)

all the above derived equations remain valid, but now with $P_{\alpha}(\alpha)$ and $P_{\alpha}^{(s)}(|\alpha|)$ in place of $P_n(\alpha)$ and $P_n^{(s)}(|\alpha|)$.

The sampling method can be used to estimate the statistical error of the reconstructed density matrix, which is quite similar to balanced homodyning [6]. Since in an experiment the number of recorded events is finite, only estimates
of the exact probabilities \(P_n\) can be obtained. Here \(\mathcal{N}_e(\varphi)\) is the total number of events at phase \(\varphi\), and \(\mathcal{N}_r(\varphi)\) is the number of events with \(n\) counts. If the measurements are performed at \(\mathcal{N}_e\) different phases \(\varphi_i\), \(i=1,2,\ldots,\mathcal{N}_e\), from Eq. (11) the estimates

\[
\langle m + s | \hat{\varrho}^{\text{[est]}} | m \rangle = \frac{1}{\mathcal{N}_e} \sum_{i=1}^{\mathcal{N}_e} \sum_{n=0}^{N} \Pi_{m,n}^{(s)} e^{is\varphi_i} P_n^{\text{[est]}}(\varphi_i)
\]

(19)

of the density-matrix elements are obtained. Note that in Eq. (19) both \(P_n^{\text{[est]}}(\varphi_i)\) and \(\langle m + s | \hat{\varrho}^{\text{[est]}} | m \rangle\) are random variables, the statistical errors of which are commonly given in terms of the square roots of the variances. Following the arguments given in Ref. [6], the quantities \(\mathcal{N}_e(\varphi)\) can approximately be regarded as independent Poissonian random variables whose means and variances are given by \(P_n^{\text{[est]}}\mathcal{N}_e = P_n^{\text{[est]}}\mathcal{N}_r\). The variance of \(P_n^{\text{[est]}}\) may then be approximated by \(P_n^{\text{[est]}}\mathcal{N}_e\), so that the variances of the real and imaginary parts of the density matrix can be estimated as

\[
\text{Var}(\text{Re}(m + s | \hat{\varrho}^{\text{[est]}} | m)) \approx \frac{1}{\mathcal{N}_e^2} \sum_{i=1}^{\mathcal{N}_e} \sum_{n=0}^{N} F_{m,n}^{(s)} \cos^2(s\varphi_i) P_n^{\text{[est]}}(\varphi_i)
\]

(20)

and

\[
\text{Var}(\text{Im}(m + s | \hat{\varrho}^{\text{[est]}} | m)) \approx \frac{1}{\mathcal{N}_e^2} \sum_{i=1}^{\mathcal{N}_e} \sum_{n=0}^{N} F_{m,n}^{(s)} \sin^2(s\varphi_i) P_n^{\text{[est]}}(\varphi_i),
\]

(21)

respectively. Thus the error can be estimated in real time as the experiment runs, simultaneously with the reconstruction of the density-matrix elements. It is worth noting that the statistical error depends on the chosen absolute value of \(|\alpha|\). For \(|\alpha|\) close to zero the diagonal density-matrix elements can be determined very precisely, whereas the off-diagonal elements strongly fluctuate. With increasing \(|\alpha|\) the precision of determining the off-diagonal elements can be improved, but the fluctuation of the diagonal elements can increase. To compensate for the fluctuation, the number of measurement events must be increased.

**FIG. 1.** Reconstruction of the density matrix of an odd coherent state \(|\psi\rangle = \mathcal{A} (\gamma \gamma^* - \gamma^* \gamma)\), \(\gamma = 1.6\). In the computer simulation \(\mathcal{N}_e = 20\) phases are considered and \(\mathcal{N}_r = 2 \times 10^5\) measurement events at each phase are assumed to be recorded using photon chopping with \(\mathcal{N}_c = 50\). The other parameters are \(|\alpha| = 1.5\), \(n_0 = 7\), \(N = 20\), and \(\eta = 0.8\). (a) Exact (real) density matrix \(|m| \hat{\varrho}^{(\text{[est]})} |n\rangle\); (b) real part of the reconstructed density matrix \(\langle m | \hat{\varrho}^{(\text{[est]})} | n \rangle\); (c) statistical error of the real part of the reconstructed density matrix calculated according to Eq. (20); (d) absolute difference between the real parts of the reconstructed and the exact density-matrix elements, \(\delta r_{n,m} = |r_{n,m}^{\text{[est]}} - r_{n,m}|\).
Another source of error stems from the truncation of the reconstructed density matrix at the value \( n_0 \). Neglecting in Eq. (7) the terms with \( m > n_0 - s \) causes a systematic error. The error can be decreased with increasing \( n_0 \). On the other hand, increasing \( n_0 \) may also cause an increase of the statistical error. This suggests that for a given number of measurement events there is some optimal value of \( n_0 \) for which the systematic error is reduced below the statistical error. The choice of such an \( n_0 \) can be done either from some \textit{a priori} information about the state or from the measured data as follows. The density matrix is reconstructed using different (probe) values of \( n_0 \). The definite value of \( n_0 \) is then chosen such that the reconstructed quantum state is normalized with a desired precision and further increase of \( n_0 \) may change the density-matrix elements only within the interval given by the statistical fluctuation [estimated according to Eqs. (20) and (21)].

To illustrate the method, we have performed computer simulations of measurements choosing the same state as in Ref. [1] and a comparable number of measurement events. The state is an odd coherent state, \( \hat{\rho} = |\alpha\rangle \langle \alpha| \), where \( |\alpha\rangle = \hat{A} (|\gamma\rangle - |\gamma\rangle) \), \( \hat{A} \) and \( |\gamma\rangle \), respectively, being a normalization constant and a coherent state \( (\gamma=1.6) \). The truncation parameter is \( n_0 = 7 \) (note that the probability for finding more than seven photons is less than 0.3%). At each phase \( N_{ev}=2 \times 10^3 \) coincident events up to \( N=20 \) are assumed to be recorded using photon chopping with \( N_c=50 \), the detection efficiency being \( \eta=0.8 \). Altogether \( N_{ev}=20 \) phases are considered choosing \( |\alpha|=1.5 \). In Fig. 1 the real parts of the exact density matrix (a) and the measured density matrix (b) are plotted (note that the imaginary part of the exact density matrix is equal to zero). As can be seen, the reconstructed density matrix is in a good agreement with the exact one. It clearly reveals the oscillation of the photon-number probability (diagonal elements) and also the phase dependence encoded in the off-diagonal elements. Further, the errors are seen to be given by the statistical fluctuations [compare the plots (c) and (d)], i.e., the errors owing to truncation of the density matrix in the reconstruction procedure are beyond the resolution limit. (If we do not know the original state, we can verify this reconstructing the density matrix with larger \( n_0 \).)

In conclusion, we have presented a method for measuring the density matrix of an optical single-mode field using unbalanced homodyning. The most important features of the method can be summarized as follows. (i) The quantum state is obtained by direct sampling of the density matrix in the photon-number basis from the measured data. This enables one to reconstruct the density matrix and to estimate the statistical error in real time. (ii) The method applies to both pure and mixed states. The density matrix of the signal mode is known when the displaced photon-number distributions (for chosen absolute value of the displacing parameter) are known, i.e., when the photon-number distribution of the interfering field is measured for a suitably large number of local-oscillator phases in a 2 \( \pi \) interval. (iii) Compared with balanced four-port homodyning [5,6] and the double-homodyne scheme [7], the discrete density matrix is obtained from discrete photon-number distributions in place of continuous field-strength distributions and phase-space functions, respectively. (iv) Whereas in balanced homodyning highly efficient linear-response photodiodes can be used, in unbalanced homodyning highly efficient detectors that distinguish between single photons must be used in order to measure photon-number distributions. Since such detectors have not been available, photon-chopping multiports that enable one to take advantage of highly efficient avalanche photodiodes may be used. (v) The method is based on the fact that the density matrix of any physical state can be truncated at a suitably large photon number \( n_0 \) —an assumption that has been made in various approaches to the problem of quantum-state measurement (see, e.g., [7,9–11]). Here, the value of \( n_0 \) is obtained either from the measurements or from some \textit{a priori} information about the state. (vi) The reconstruction is performed using the method of least squares, which is a very powerful tool also in other branches of physics. Here it enables one to find the density matrix that best fits the measured data.

Note added in proof. After preparing the paper we were made aware of a paper on reconstruction of quantum states of trapped atoms [12] with a mathematically similar formalism.

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