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Number–phase uncertainty relations

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Abstract. The minimization problem of finding the number–phase minimum uncertainty states (MUS) is considered and its solutions are found either numerically or, under some special conditions, analytically. The phase uncertainty measure is based on the Bandilla–Paul dispersion. The problem is treated (i) in a finite-dimensional Hilbert space and (ii) for a countably infinite-dimensional Hilbert space (i.e. the standard quantum harmonic oscillator), with the constraint of a given mean photon number. The MUS relations between the photon number uncertainty and phase uncertainty are presented. Connections to some other minimization problems are discussed.

1. Introduction

The uncertainty principle and uncertainty relations are among the central and most fundamental concepts of quantum theory. The presence of some limitations invoke questions such as: where is the boundary that the quantum world does not allow us to overcome? What are the states that reach this boundary? This paper addresses these questions with regards to the variables of quantum phase and photon number.

The fact that minimizing the spread of photon (or phonon) number distribution of a quantum oscillator causes loss of phase information and vice versa was clear from the early days of quantum mechanics. Dirac was the first to try to quantify this by an uncertainty relation

$$\Delta\phi\Delta n \geqslant \tfrac{1}{2} \tag{1}$$

in analogy to the well known position–momentum uncertainty relation. However, Dirac's relation is problematic because it requires $\Delta\phi$ to be larger than π for sufficiently small Δn. This relation was derived from an incorrect assumption that there exists a Hermitian phase operator $\hat{\phi}$ conjugated to the photon number operator $\hat{n}$. Further investigations [1–3] have shown that introducing operators referring to phase requires a much more sophisticated approach, which then leads to more complicated uncertainty relations. For example, the Susskind–Glogower [1] cosine $\hat{C}$ and sine $\hat{S}$ operators fulfilling the commutation relations

$$[\hat{n},\hat{S}] = \mathrm{i}\hat{C} \qquad [\hat{n},\hat{C}] = -\mathrm{i}\hat{S} \tag{2}$$

imply uncertainty relations

$$\langle(\Delta n)^2\rangle\langle(\Delta C)^2\rangle \geqslant \frac{\langle S\rangle^2}{4} \qquad \langle(\Delta n)^2\rangle\langle(\Delta S)^2\rangle \geqslant \frac{\langle C\rangle^2}{4} \tag{3}$$

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or a symmetric relation [3]

$$\left[\langle(\Delta n)^2\rangle + \tfrac{1}{4}\right]\left[\langle(\Delta C)^2\rangle + \langle(\Delta S)^2\rangle + \tfrac{1}{2}P_0\right] \geqslant \tfrac{1}{4} \tag{4}$$

where P_0 is the probability of finding the oscillator in the ground state (arising from the non-commuting nature of $\hat{C}$ and $\hat{S}$). Also, in the Pegg–Barnett model [4–6], which postulates a finite-dimensional Hilbert space, we can get a state-dependent uncertainty relation [7] due to the non-trivial nature of the number–phase commutator. Of course, neither of these relations gives a simple answer to the question: given a fixed photon number spread, what is the minimum phase uncertainty that quantum states can have?

Before trying to answer this question we should be clear about how to define the uncertainty. A possible way, usual in the case of Euclidean variables is to take the standard deviation, i.e. square root of variance, $\Delta_\theta \phi = \sqrt{D_\theta(\phi)}$, where

$$D_\theta(\phi) = \int_\theta^{\theta+2\pi} (\phi - \langle\phi\rangle_\theta)^2 p(\phi)\,\mathrm{d}\phi \tag{5}$$

and where the mean is

$$\langle\phi\rangle_\theta = \int_\theta^{\theta+2\pi} \phi p(\phi)\,\mathrm{d}\phi\,. \tag{6}$$

It has the advantage that for a pair of non-commuting variables an uncertainty relation can be written for the product of uncertainties—following from the Cauchy–Schwarz inequality. However, such uncertainty relations are often state dependent and it is difficult to present a universal limit, valid for all states. There is also a reason against using the variance-based measure of phase uncertainty following from its type of ambiguity: it is well known that changing the phase window $[\theta, \theta + 2\pi)$ over which we calculate means generally a change in the mean (6) and the variance (5) [8]. To overcome this disadvantage and to benefit from other properties, various other phase uncertainty measures were suggested, for their review see for example, [9, 10]. A useful approach to this problem is based on integrating periodic functions of phase rather than the phase itself. We can calculate the mean of the exponential of phase and write it in the goniometric form

$$\langle \mathrm{e}^{\mathrm{i}\phi}\rangle = R_\phi \mathrm{e}^{\mathrm{i}\bar{\phi}} \tag{7}$$

where $\bar{\phi}$ is a uniquely defined phase mean and R_ϕ is connected to the *dispersion* σ_ϕ^2, a measure of phase uncertainty introduced by Bandilla and Paul [11]. Here the mean is calculated using the phase distribution

$$\langle \mathrm{e}^{\mathrm{i}\phi}\rangle = \int_0^{2\pi} \mathrm{e}^{\mathrm{i}\phi} p(\phi)\,\mathrm{d}\phi\,. \tag{8}$$

The phase distribution $p(\phi)$ generally depends on the measurement scheme. In this work we will consider the canonical phase distribution, i.e. for a state $|\psi\rangle$ the phase distribution is $p(\phi) = \frac{1}{2\pi}|\langle\phi|\psi\rangle|^2$ and $|\phi\rangle = \sum_n \exp(\mathrm{i}n\phi)|n\rangle$, which is equivalent to calculation of the mean using the Susskind–Glogower operators as

$$\langle \mathrm{e}^{\mathrm{i}\phi}\rangle = \langle\psi|(\hat{C} + \mathrm{i}\hat{S})|\psi\rangle\,. \tag{9}$$

(Other phase-measurement statistics, which may be especially interesting from the experimental point of view are not considered here. For a review see, e.g. [12–14].) The dispersion is then defined as

$$\sigma_\phi^2 = 1 - R_\phi^2 \tag{10}$$

usually it is calculated as

$$\sigma_\phi^2 = 1 - \langle C\rangle^2 - \langle S\rangle^2\,. \tag{11}$$

This measure of uncertainty takes values between zero and unity and is uniquely defined. As may be shown, for a sharply peaked phase distributions and for a properly chosen phase window, the variance gives similar results as the dispersion. An uncertainty relation containing the dispersion can be derived from (3) [8]:

$$\left(\langle(\Delta\hat{n})^2\rangle + \tfrac{1}{4}\right)\sigma_\phi^2 \geqslant \tfrac{1}{4}\,. \tag{12}$$

In this paper we will use a phase uncertainty measure $\Delta\phi$ based on the dispersion and defined as [15]

$$\Delta\phi = \arcsin\sigma_\phi\,. \tag{13}$$

Similarly to dispersion this measure is uniquely defined; it takes values between zero and $\pi/2$ and for sharply peaked distributions it yields similar results to the square root of variance. In contrast to the dispersion it is measured in radians—and as discussed in [15] has a simple physical interpretation based on the analogy between probabilistic and mechanical quantities.

Let us briefly mention this point, treated in detail in [15]. As is well understood, the mechanical analogue of the mean value is the centre of mass, and similarly the analogue of the dispersion is the moment of inertia with respect to the centre of mass. Suppose a one-dimensional body with unit mass, described by the mass density $p(x)$; considering only translational motion we can substitute for this body by a unit mass point located at the centre of mass $\langle x\rangle$. If we consider both translational and rotational motion of the body, we can substitute it by two mass points (each with mass $\frac{1}{2}$) located at $\langle x\rangle \pm \sqrt{D(x)}$, where $D(x) = \int(x - \langle x\rangle)^2 p(x)\,\mathrm{d}x$ is the moment of inertia. Continuing this analogy to the phase variable we can imagine a ring with unit radius and unit mass, its mass density being described by $p(\phi)$. The centre of mass of this ring has polar coordinates R_ϕ and $\bar{\phi}$ given by (7), and (8). The moment of inertia with respect to the centre of mass is given by the dispersion σ_ϕ^2 (10) and the rotational properties of the ring (about axes perpendicular to the plane of the ring) are equivalent to those of two one-half mass points located on the ring in positions $\bar{\phi} \pm \Delta\phi$, where $\Delta\phi$ is given by (13). Thus using the window-dependent mean (6) and variance (5) for describing phase properties corresponds to 'cutting' the ring at some point θ, making it straight and then finding its centre of mass and moment of inertia. As a contrast, using the quantities $\bar{\phi}$, σ_ϕ^2 and $\Delta\phi$ keeps the circular shape of the ring when describing its properties. This may be said as a response to Hall [9] who lacks a physical interpretation of the Bandilla–Paul dispersion.

For this measure the Chebyshev inequality can be obtained in the form [15]

$$P(|\phi - \bar{\phi}| > \epsilon) \leqslant \frac{\sin^2(\Delta\phi)}{1 + \cos^2(\Delta\phi) - 2\cos(\Delta\phi)\cos\epsilon} = \frac{1 - R_\phi^2}{1 + R_\phi^2 - 2R_\phi\cos\epsilon} \tag{14}$$

which connects the uncertainty as a measure of width with the probability to be in an interval centred in the mean value (here $\epsilon \in (0, \pi)$ and P means the probability of the event in the parentheses). Note that the right-hand side of equation (14) is the well known Poisson kernel.

An uncertainty relation, equivalent to (12), can be obtained for this uncertainty measure in the form

$$\Delta n \tan\Delta\phi \geqslant \tfrac{1}{2}\,. \tag{15}$$

However, this relation (similarly to (12)) is too weak, which means that it allows wider class of points $(\Delta n, \Delta\phi)$ than is actually possible; e.g. there are no states for which the equality holds. There must be some stronger limitation and it is the aim of this paper to find it—together with the states that reach the limits.

The paper is organized as follows. In the next section known methods for finding the minimum uncertainty states are discussed and a variant of one of the methods is presented, suitable for solving the above problems. Section 3 is devoted to finding the uncertainty relation in the finite-dimensional Hilbert spaces and then in the limit $N \to \infty$, i.e. in the case of the Pegg–Barnett model. The last section discusses the uncertainty relations in the case when a mean photon number is given.

2. A method for finding minimum uncertainty states

There are two usual methods for finding the (variance-based) minimum uncertainty states (MUS); both assume the uncertainty relation in the form where the product of the variances is greater than some limit [16, 2]. (In this paper we use the term MUS for every state that minimizes the uncertainty of one quantity when the uncertainty of the other quantity is given; in the terminology introduced by Aragone *et al* [17] such states would be called *intelligent* states, the term MUS being reserved for states minimizing the product of the uncertainties.) Let us consider two non-commuting quantities $\hat{x}$, $\hat{y}$, then

$$\langle(\Delta\hat{x})^2\rangle\langle(\Delta\hat{y})^2\rangle \geqslant \tfrac{1}{4}|\langle[\hat{x},\hat{y}]\rangle|^2 + \tfrac{1}{4}|\langle\{\Delta\hat{x},\Delta\hat{y}\}\rangle|^2 \tag{16}$$

where $\Delta\hat{x} \equiv \hat{x} - \langle x\rangle$ and $\Delta\hat{y} \equiv \hat{y} - \langle y\rangle$. The MUSs, with respect to the relation (16), are such states for which the mean anticommutator (i.e. quantum covariance) vanishes, $\langle\{\Delta\hat{x},\Delta\hat{y}\}\rangle = 0$, and the inequality turns into an equality. The *direct method* assumes that the commutator is a c-number; the equality in (16) then appears for such states $|\psi\rangle$ for which $\Delta\hat{x}|\psi\rangle = c\Delta\hat{y}|\psi\rangle$, where c is a constant. The condition of zero mean anticommutator implies that $c = -\mathrm{i}\gamma$, where γ is real. Thus these requirements lead to the eigenvector equation

$$(\Delta\hat{x} + \mathrm{i}\gamma\Delta\hat{y})|\psi\rangle = 0\,. \tag{17}$$

There are three real parameters here: $\langle x\rangle$, $\langle y\rangle$ and γ. Changing γ and solving (17) we obtain various MUSs with the means $\langle x\rangle$ and $\langle y\rangle$.

For quantities whose commutator is a q-number and which then yield state-dependent uncertainty relations, Jackiw [16] derived an *analytic method* for finding MUS. Setting the variation of the uncertainty product to be zero, $\delta(\langle(\Delta\hat{x})^2\rangle\langle(\Delta\hat{y})^2\rangle) = 0$, he obtained an Euler–Lagrange equation for $|\psi\rangle$ (actually an eigenvector equation):

$$\left[\frac{\Delta\hat{x}^2}{\langle(\Delta\hat{x})^2\rangle} + \frac{\Delta\hat{y}^2}{\langle(\Delta\hat{y})^2\rangle} - 2\right]|\psi\rangle = 0\,. \tag{18}$$

Here we have four real parameters: $\langle x\rangle$, $\langle y\rangle$, $\langle(\Delta x)^2\rangle$ and $\langle(\Delta y)^2\rangle$, the last two being connected by the uncertainty relation with equality sign. Solutions of (18) are stationary states of the uncertainty product; among them we have to choose the one with the minimum product.

It may be worth noting that the coherent states and the two-photon coherent (squeezed) states, which are MUSs with respect to x and p, can be identified either as eigenstates of some generalized annihilation operators (as in the direct method) or as ground states of harmonic oscillator Hamiltonians (as in the analytic method).

The method used here is essentially based on the Jackiw analytical method: we will seek for ground states of some operators, which we call 'uncertainty Hamiltonians'. The main idea is as follows. Let, under some constraints, the uncertainty Δq of a quantity q be defined as $\Delta q \equiv f_q(\langle\hat{B}_q\rangle)$, where f_q is some increasing function and $\hat{B}_q$ is a Hermitian operator with spectrum limited from below. (As an example we can take as the position uncertainty the function $f_x(\langle \hat{x}^2 \rangle) = \sqrt{\langle \hat{x}^2 \rangle}$ under the constraint that $\langle x \rangle = 0$, or as the phase uncertainty the function $f_\phi(\langle -\hat{C} \rangle) = -\arccos\langle -\hat{C} \rangle$ under the constraint of zero mean phase.) The constraints here play the role of parameters in the above methods. The minimum uncertainty states with respect to a pair of quantities q and s then can be found as the ground states $|\psi_\xi\rangle$ of the uncertainty Hamiltonians

$$\hat{H}_{unc}(\xi) \equiv \xi \hat{B}_q + (1 - \xi)\hat{B}_s \tag{19}$$

where ξ is a parameter between zero and unity. The uncertainty relation for the MUSs can then be written as a parametric equation

$$\Delta q_{(\xi)} = f_q(\langle \psi_\xi | \hat{B}_q | \psi_\xi \rangle) \qquad \Delta s_{(\xi)} = f_s(\langle \psi_\xi | \hat{B}_s | \psi_\xi \rangle) . \tag{20}$$

The uncertainty relation then says that no state with uncertainty $\Delta s = \Delta s_{(\xi)}$ for some ξ can have less value of Δq than $\Delta q_{(\xi)}$. The proof is straightforward: suppose such a state $|\psi_?\rangle$ for which $\Delta s = \Delta s_{(\xi)}$ and $\Delta q < \Delta q_{(\xi)}$. Then $\langle \psi_? | \hat{B}_s | \psi_? \rangle = \langle \psi_\xi | \hat{B}_s | \psi_\xi \rangle$ and $\langle \psi_? | \hat{B}_q | \psi_? \rangle < \langle \psi_\xi | \hat{B}_q | \psi_\xi \rangle$, due to increasing of f_q. For such state it would then be $\langle \psi_? | \hat{H}_{unc}(\xi) | \psi_? \rangle < \langle \psi_\xi | \hat{H}_{unc} | \psi_\xi \rangle$ which is a contradiction of the assumption that $|\psi_\xi\rangle$ is the ground state.

This method may be easily generalized for finding relations among higher numbers of positive-definite quantities; the Hamiltonian (19) would then depend on more parameters.

A very similar method for finding various extremal states was also used by other authors [18–24], however, their idea was based on solving extremum problems using Lagrange multipliers. Also the conditions for the extrema were not the same in these works. We will discuss some of their results with respect to the problems presented here.

Solutions of the ground-state problems of the Hamiltonians (19) can easily be found numerically with arbitrary precision if we express the Hamiltonian in the Fock basis and truncate the expansion at a sufficiently large photon number. Another possible way is to work in a finite-dimensional Hilbert space and observe the behaviour of the solutions when the dimension tends to infinity, as is the idea of the Pegg–Barnett model. In both cases we can use standard software routines for finding eigenvalues and eigenvectors of matrices. It is also possible to work with familiar approximate methods for solving ground-state problems, like the perturbation method or the Ritz variational method.

3. Number–phase uncertainty relations in finite-dimensional Hilbert spaces

We will first use the method of the uncertainty Hamiltonian for finding the MUS in a finite-dimensional Hilbert space. Let us consider such an N-dimensional space to be spanned by eigenvectors of two complementary operators $\hat{n}$ and $\hat{\phi}$. The interpretation of these operators is quite arbitrary; however, finally in the limit $N \to \infty$ we will treat them as the Pegg–Barnett operators of photon number and phase. Another possible interpretation, perhaps better than number and phase in the case of N finite, is connected to a model of a particle moving along a circle with a finite number of sites: one quantity would refer to the particle's position, whereas the other to its discrete (angular) momentum. The eigenvectors of these operators are connected by the discrete Fourier transform

$$\begin{aligned} |\phi_k\rangle &= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \exp\left(\mathrm{i}\frac{2\pi}{N} kn \right) |n\rangle \\ |n\rangle &= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \exp\left(-\mathrm{i}\frac{2\pi}{N} nk \right) |\phi_k\rangle . \end{aligned} \tag{21}$$

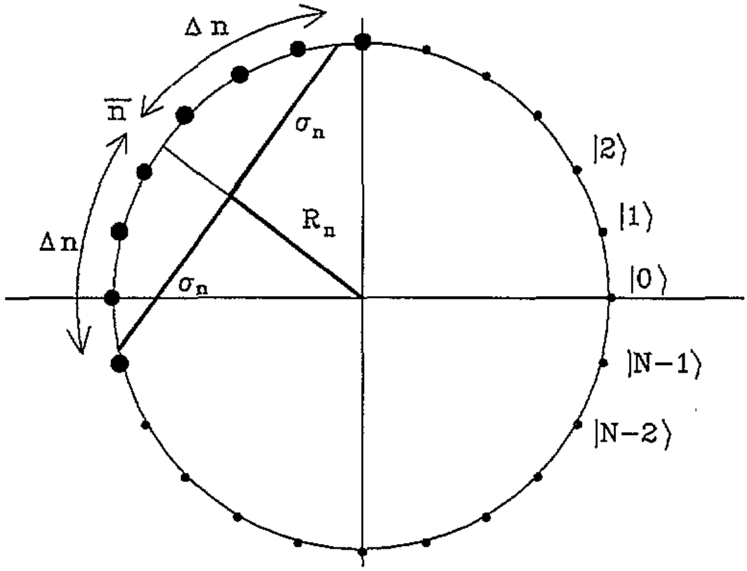

Figure 1. 'Number uncertainty' in a finite-dimensional Hilbert space. The definitions (22)–(25) are with respect to the circular topology of the space.

(These relations in this or similar form are discussed either in the original papers on the Pegg–Barnett formalism, e.g. [4–6], or in papers dealing with a dynamics in finite-dimensional Hilbert spaces, e.g. [25, 26].) We can notice the natural topology of such a model—after the last eigenstate ($|\phi_{N-1}\rangle$ or $|N-1\rangle$) the first one ($|\phi_0\rangle$ or $|0\rangle$) follows. It suggests that for measuring uncertainties of such quantities we should use a similar measure as for angular or phase variables—here we will work with the measure based on (11) and (13). Let us call 'normalized dispersions' the quantities σ_ϕ^2 and σ_n^2 calculated as

$$\sigma_n^2 = 1 - R_n^2 \qquad \sigma_\phi^2 = 1 - R_\phi^2 \tag{22}$$

where

$$\begin{aligned} R_n^2 &= \left(\sum_n p_n(n)\cos\left(\frac{2\pi}{N}n\right)\right)^2 + \left(\sum_n p_n(n)\sin\left(\frac{2\pi}{N}n\right)\right)^2 \\ R_\phi^2 &= \left(\sum_k p_\phi(\phi_k)\cos\left(\frac{2\pi}{N}k\right)\right)^2 + \left(\sum_k p_\phi(\phi_k)\sin\left(\frac{2\pi}{N}k\right)\right)^2 . \end{aligned} \tag{23}$$

Here for a state given by the density operator $\hat{\rho}$ the probabilities are

$$p_n(n) = \langle n|\hat{\rho}|n\rangle \qquad p_\phi(\phi_k) = \langle\phi_k|\hat{\rho}|\phi_k\rangle . \tag{24}$$

Similarly to the Bandilla–Paul dispersion these normalized dispersions take values between zero (for a sharp value of n or ϕ) and unity (e.g. for uniformly spread probabilities). If we want to interpret $\hat{\phi}$ as phase and $\hat{n}$ as photon number, it is better to measure the ϕ uncertainty in radians and the n uncertainty in numbers which increase with increasing N. For the n uncertainty we also require that it yields the same results as the standard deviation in the limit $N \to \infty$. Thus it is natural to define the ϕ uncertainty $\Delta\phi$ as in equation (13) and the n uncertainty Δn as (see figure 1)

$$\Delta n = \frac{N}{2\pi} \arcsin \sigma_n . \tag{25}$$

The last measure takes values between zero and $N/4$. Here the mean $\bar{n}$ is defined by

$$R_n \exp\left(\mathrm{i}\frac{2\pi}{N}\bar{n}\right) = \sum_n p_n(n) \exp\left(\mathrm{i}\frac{2\pi}{N}n\right). \tag{26}$$

As may easily be checked, for states with excited $|n\rangle$ components only with $n \ll N$ (condition of the Pegg–Barnett model) does our definition of Δn give almost the same results as the standard deviation $\sqrt{\langle n^2\rangle - \langle n\rangle^2}$, and the mean $\bar{n}$ is very close to the usual mean $\langle n\rangle$, where $\langle n\rangle = \sum_n p_n(n)n$. On the other hand, our definition respects the circular topology of the variable: whereas the standard deviation would give for the superposition of the extreme states $(1/\sqrt{2})(|0\rangle + |N-1\rangle)$ the value $(N-1)/2$, our uncertainty is $\Delta n = \frac{1}{2}$.

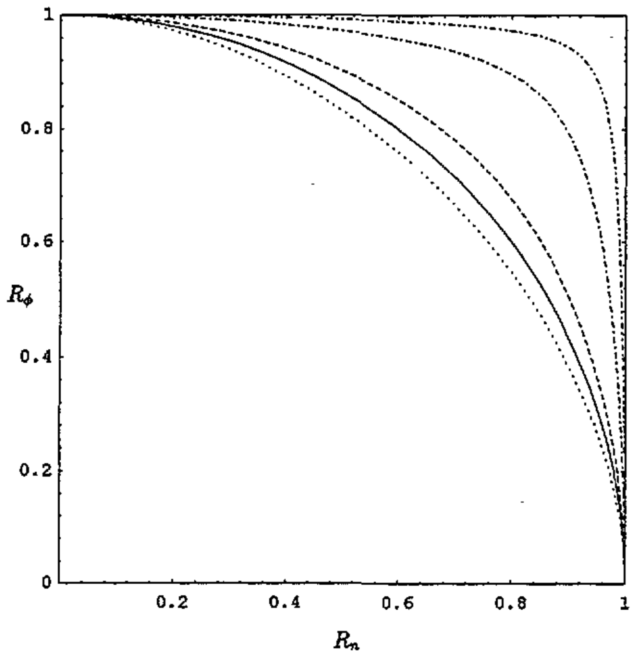

Figure 2. Dependence between the MUS parameters R_n and R_ϕ for various dimensions of the Hilbert space. $N = 2$, $N = 4$ (——); $N = 3$ (······); $N = 5$ (– – –); $N = 10$ (— · —); $N = 20$ (— · · —).

Now let us turn our attention to find the MUS. We have to construct the uncertainty Hamiltonian (19) and to choose conditions for means $\bar{\phi}$ and $\bar{n}$. A simple choice is to require that

$$\bar{\phi} = 0 \qquad \text{and} \qquad \bar{n} = 0 \tag{27}$$

the same final results for uncertainties would also be obtained for other integer $\bar{n}$ and integer k, $\bar{\phi} = \frac{2\pi}{N}k$. Then $R_n = \langle \hat{C}_n\rangle$ and $R_\phi = \langle \hat{C}_\phi\rangle$, where

$$\hat{C}_n = \sum_n |n\rangle \cos\left(\frac{2\pi}{N}n\right)\langle n| \qquad \hat{C}_\phi = \sum_k |\phi_k\rangle \cos\left(\frac{2\pi}{N}k\right)\langle \phi_k| . \tag{28}$$

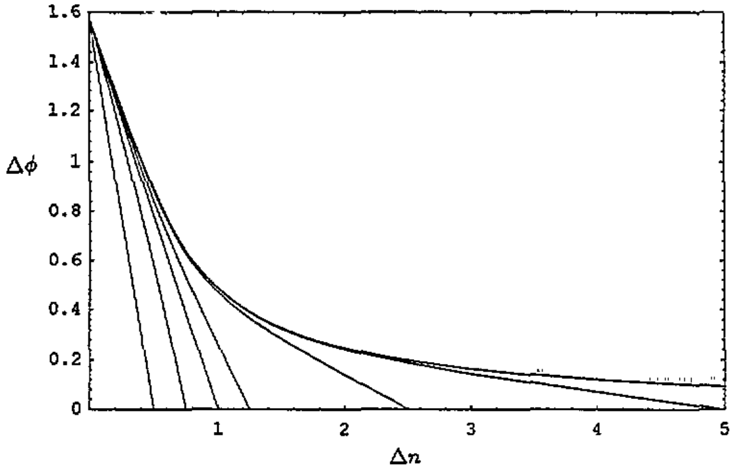

Figure 3. Relation between the MUS uncertainties Δn and $\Delta\phi$ for various dimensions of the Hilbert space. The curves ending at the horizontal axis are successively for $N = 2, 3, 4, 5, 10, 20, 40$.

In the n basis these operators have the expansions

$$\langle m|\hat{C}_n|r\rangle = \delta_{m,r} \cos\left(\frac{2\pi}{N}m\right) \qquad \langle m|\hat{C}_\phi|r\rangle = \tfrac{1}{2}(\delta_{m,r+1} + \delta_{m+1,r}) \tag{29}$$

where the addition of indices is taken modulo N. To ensure the condition (27), we write in the uncertainty Hamiltonian $\hat{B}_n \equiv -\hat{C}_n$ and $\hat{B}_\phi \equiv -\hat{C}_\phi$ so that

$$\hat{H}(\xi) = -\xi \hat{C}_n - (1-\xi)\hat{C}_\phi\,. \tag{30}$$

Changing the parameter ξ between zero and unity and finding the ground states of the Hamiltonian we obtain all the MUSs fulfilling the condition (27) (particularly for $\xi = 0$ we obtain the $|\phi_0\rangle$ state and for $\xi = 1$ the $|0\rangle$ state). We can calculate for these states the means $\langle\hat{C}_n\rangle$ and $\langle\hat{C}_\phi\rangle$ and from them the uncertainties $\Delta\phi$ and Δn. Let us mention that ground states of the uncertainty Hamiltonian (30) with $\xi = \frac{1}{2}$ have been used in the definition of a discrete Q-function in [27].

The ground-state eigenvalue equation can be solved analytically for the lowest values of N; for higher values we can get the results numerically. Thus for $N = 2, 3, 4$ we obtain the MUS relation between R_n and R_ϕ in the form

$$R_n^2 + R_\phi^2 = 1 \qquad (N = 2, 4) \tag{31}$$

and

$$3R_n^2 + 3R_\phi^2 + 2R_nR_\phi - 2R_n - 2R_\phi = 1 \qquad (N = 3)\,. \tag{32}$$

These results together with the MUS relations between R_n and R_ϕ for higher N are depicted in figure 2. All states allowed by the quantum theory have their R_n, R_ϕ parameters below this curve, i.e. with R_n and R_ϕ equal to or less than the values given by these relations. For $N \geqslant 3$ the area of allowed R parameters is increasing with increasing N; it is interesting that for $N = 3$ we have a stronger constraint than for $N = 2$.

Considering the relations between uncertainties Δn and $\Delta\phi$ we get the results given in figure 3. For $N = 2$ and $N = 4$ we obtain simple results in the form of linear uncertainty relations:

$$\begin{aligned} \Delta\phi + \pi\Delta n &\geqslant \frac{\pi}{2} \qquad (N=2) \\ \Delta\phi + \frac{\pi}{2}\Delta n &\geqslant \frac{\pi}{2} \qquad (N=4). \end{aligned} \tag{33}$$

With N increasing we see that the Δn, $\Delta\phi$ MUS dependence approaches a limit—expressing the $n \sim \phi$ uncertainty relation in the Pegg–Barnett model. (For comparison of the $\Delta n \sim \Delta\phi$ relation of the Pegg–Barnett model with the Dirac relation (1) and the relation (15) see figure 4.) However, it should be kept in mind that in the Pegg–Barnett model the states cannot have components 'from the opposite side of zero', i.e. with $n \approx N$. Therefore the states approaching the uncertainty limit must have sufficiently large mean photon number, $\bar{n} \approx \langle n \rangle \gg \Delta n$.

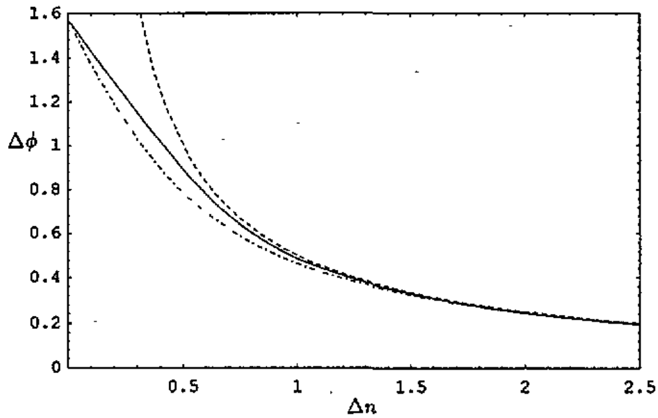

Figure 4. Comparison between various $\Delta n \sim \Delta\phi$ uncertainty relations: the Dirac relation (1) (- - -), the weak relation (15) (— · —), and the limiting relation for $N \to \infty$ (——).

4. Uncertainty relations including mean photon number

The limiting results ($N \to \infty$) from the last section are applicable for describing the $n \sim \phi$ uncertainties only for sufficiently high mean photon numbers, when the vacuum state $|0\rangle$ lies far outside the interval $(\langle n\rangle - \Delta n;\ \langle n\rangle + \Delta n)$. However, in real situations the photon number uncertainty may be of the same order as the mean; therefore it makes sense to ask what is the form of the $n \sim \phi$ uncertainty relation under the condition that the mean photon number is given. It is clear that in this case the phase uncertainty cannot be arbitrarily small, even though we infinitely increase the photon number uncertainty. Recently, interesting extremization problems related to this problem were studied, namely the problem of finding states with a given finite mean photon number and minimizing phase dispersion [21, 28, 29], minimizing phase variance [19], minimizing the photon number operator and the quadrature operator [20], minimizing the photon number uncertainty and the sine or cosine uncertainties [22] or minimizing the phase dispersion and the angular momentum uncertainty of a plane rotator [23]. In this section we will first discuss this limit of phase uncertainty given by the finite mean phase number and compare the results with phase uncertainties of several important states. Then we will find the relation between $\Delta\phi$ and Δn in the limit of $\langle n\rangle \to \infty$ and finally we will consider the most general case for arbitrary $\langle n\rangle$.

4.1. Phase uncertainty versus mean photon number

Let us now briefly recapitulate the results of minimizing the phase dispersion and the photon number from the point of view of our method. The minimizing states (with $\bar{\phi} = 0$) are the ground states of the uncertainty Hamiltonian

$$\hat{H}(\xi) = \xi\hat{n} - (1-\xi)\hat{C} \tag{34}$$

where $\hat{C}$ is the Susskind–Glogower cosine operator. Here $\hat{B}_\phi \equiv -\hat{C}$, to ensure that $\bar{\phi} = 0$. If we express this Hamiltonian in the Fock basis and look for the eigenstates in the form

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle \tag{35}$$

we get from the eigenstate equation

$$\hat{H}(\xi)|\psi\rangle = \lambda|\psi\rangle \tag{36}$$

an infinite set of equations for the c_n coefficients

$$c_{n+1} + c_{n-1} = \left(-\frac{2\lambda}{1-\xi} + \frac{2\xi}{1-\xi}n\right)c_n . \tag{37}$$

These equations were obtained by Bandilla *et al* [21] (using the method of Lagrange multipliers) and solved analytically giving

$$c_n = \text{constant} \times J_{p+1+n}(j_{p,1}) \tag{38}$$

where J is the Bessel function, $p = -1 - \frac{\lambda}{1-\xi} j_{p,1}$ and $j_{p,1}$ is the first zero point of the Bessel function J_p. Our numerical results can be seen in figure 5, where the dependence between $\Delta\phi$ and $\langle n\rangle$ for these phase optimized states is depicted.

For comparison, the $\Delta\phi \sim \langle n\rangle$ dependence is depicted also for other important states: the coherent states, the truncated phase states

$$|\phi\rangle_s \equiv \frac{1}{\sqrt{s+1}} \sum_{n=0}^{s} e^{in\phi}|n\rangle \tag{39}$$

and the 'coherent phase states' (see, e.g. [30–32])

$$|z\rangle = \sqrt{1-|z|^2} \sum_{n=0}^{\infty} z^n|n\rangle \tag{40}$$

where $|z| < 1$. For the last two states we can easily find an analytical $\Delta\phi \sim \langle n\rangle$ dependence:

$$\Delta\phi = \arccos \frac{2\langle n\rangle}{2\langle n\rangle + 1} \tag{41}$$

for the truncated phase states and

$$\Delta\phi = \operatorname{arccot}\sqrt{\langle n\rangle} \tag{42}$$

for the coherent phase states. The phase uncertainty of coherent states was discussed in [21]. It is interesting that the coherent states have a phase uncertainty (with the same $\langle n\rangle$) smaller than both the truncated phase states and the coherent phase states which, in the limit of $s \to \infty$ or $|z| \to 1$ approach the phase states. For $\langle n\rangle \gg 1$ the phase uncertainty of coherent states behaves as $\Delta\phi \approx 1/(2\sqrt{\langle n\rangle})$, whereas for the truncated and the coherent phase states as $\Delta\phi \approx 1/\sqrt{\langle n\rangle}$, i.e. it is twice as large as the value of coherent states. The asymptotic behaviour of the phase optimized states (ground states of the Hamiltonian (34)) was found in [21] and can be expressed as $\Delta\phi \approx \sqrt{1.8936/\langle n\rangle}$.

An interesting question concerns the possibility of minimizing the mean photon number and the phase uncertainty for some given class of states. For example, Freyberger and Schleich [29] discussed such minimization for the squeezed states (two-photon states). From the point of view of the minimum uncertainty Hamiltonian method, solving this problem requires just using the Ritz variational method. Let us consider the squeezed states $|\alpha, \zeta\rangle$ [33],

$$|\alpha, \zeta\rangle \equiv \hat{D}(\alpha)\hat{S}(\zeta)|0\rangle \tag{43}$$

where $\hat{D}(\alpha) = \exp(\alpha\hat{a}^{+} - \alpha^{*}\hat{a})$ is the displacement operator and $\hat{S}(\zeta) = \exp(\zeta^{*}\hat{a}^{2}/2 - \zeta\hat{a}^{+2}/2)$ is the squeeze operator. Let the parameters α and ζ be real, $\alpha \geqslant 0$, so that we have a two-parametric class of states. The mean photon number for such states is $\langle n\rangle = |\alpha|^2 + \sinh^2|\zeta|$, then, given $\langle n\rangle$ we have a one-parametric set of states $\hat{D}(\sqrt{\langle n\rangle - \sinh^2\zeta})\hat{S}(\zeta)|0\rangle$. We can find for which values of the squeezing parameter ζ the mean value of the uncertainty Hamiltonian is minimized and for this state then calculate the phase uncertainty $\Delta\phi$. The results obtained by this approximative method for not too high $\langle n\rangle$ are in a very good agreement with the precise values—the relative difference between $\Delta\phi$ of the optimally squeezed states and $\Delta\phi$ of the actual phase optimized states is at most about 0.2% for $\langle n\rangle \leqslant 10$ and it is still less than 1% for $\langle n\rangle \leqslant 25$. (Therefore in figure 5 the curve related to the optimally squeezed states is indistinguishable from the phase optimized states curve.) The problem of minimizing $\Delta\phi$ with given $\langle n\rangle$ for squeezed states was approximately solved requiring that the contour ellipse of such state touches the origin in the phase space [29], i.e. $\alpha = \frac{1}{2}\exp(-\zeta)$.

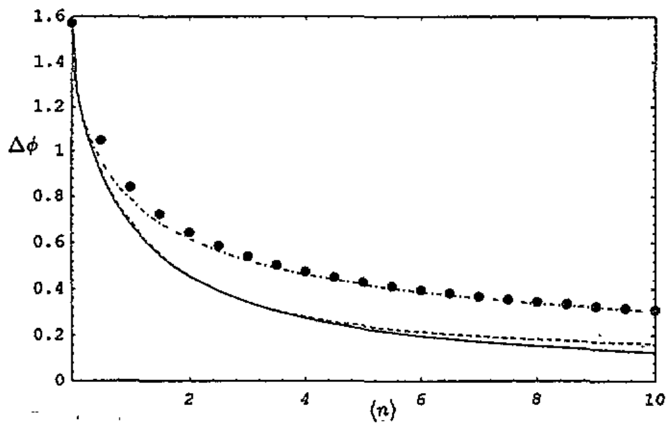

Figure 5. Phase uncertainty for several states in dependence on the mean photon number. The phase optimized states and the optimal squeezed states are indistinguishable here (——), (– – –): coherent states, (— · —): coherent phase states, (· · · · · ·): truncated phase states.

Freyberger and Schleich [29] suggested that a better assumption is to require that the squeezed state should be displaced more, i.e. $\alpha > \frac{1}{2}\exp(-\zeta)$, so as to minimize the overlap with the vacuum state. Here we can give more accurate results, valid for arbitrary values of the parameters (Freyberger and Schleich assumed $\exp(-\zeta) \gg 1$); the observed behaviour can be seen in figure 6. For very small $\langle n\rangle$ the origin can be inside the contour ellipse—the limiting case is the vacuum state. Increasing $\langle n\rangle$ from zero we obtain a slightly squeezed state, but initially with $\exp(-\zeta) < 1$, i.e. the state is stretched in the p-direction. This surprising behaviour can be explained as following from the minimization of the overlap

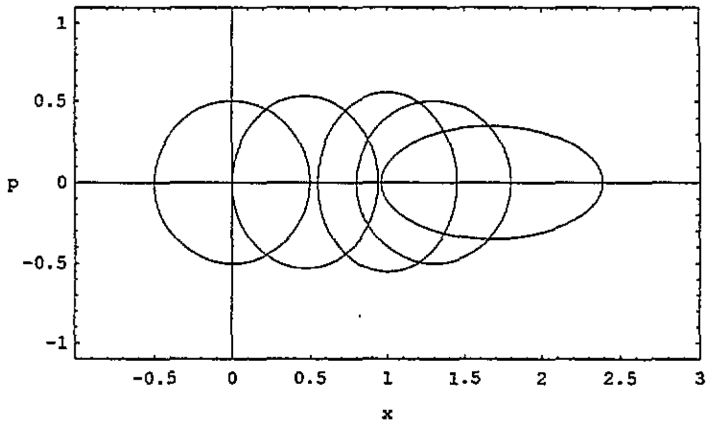

Figure 6. Optimally squeezed states with very small mean photon number. The states start from the vacuum, then when shifted along the x-axis they are first slightly stretched in the p-direction (i.e. x-squeezed). For $\langle n\rangle \approx 0.2278$ the contour curve touches the origin; for $\langle n\rangle \approx 2.605$ the state is a coherent state and for larger $\langle n\rangle$ it continues as stretched in the x-direction (p-squeezed).

with vacuum. Further increasing $\langle n\rangle$, the ζ parameter returns to zero (for $\langle n\rangle \approx 2.605$ we obtain a coherent state) and then the state is stretched in the x-direction as ζ decreases into negative values. The 'touching the origin' condition $\alpha = \frac{1}{2}\exp(-\zeta)$ is achieved for $\langle n\rangle \approx 0.2278$; for higher $\langle n\rangle$ the contour ellipse is actually more displaced, as considered in [29].

4.2. *Phase uncertainty versus number uncertainty; infinite mean photon number*

Let us now study the problem of minimizing the photon number uncertainty and the phase uncertainty for the case of a very large integer $\langle n\rangle$. If we are working with Δn sufficiently small, i.e. $\Delta n \ll \langle n\rangle$, we can find the MUS as ground states of the uncertainty Hamiltonian

$$\hat{H}(\xi) = \xi\hat{\mathcal{N}}^2 - (1-\xi)\hat{C} \tag{44}$$

where $\hat{\mathcal{N}} \equiv \hat{n} - \langle n\rangle$. Our approximation will be based on the assumption that the eigenvalues of $\hat{\mathcal{N}}$ are all integers from $-\infty$ to $+\infty$, i.e. we can treat it as the angular momentum operator $\hat{L}_z$ of a plane rotator. Such a problem was treated in [23]. In this case the Hamiltonian (44) is proportional to the Hamiltonian of a pendulum

$$\hat{H}_{pend} = \frac{\hat{L}_z^2}{2mr^2} - mgr\widehat{\cos\phi} \tag{45}$$

a mass point m constrained to move on a vertical circle with a radius r, g being the gravitational acceleration. The eigenfunctions of such Hamiltonian (in the ϕ representation) are the Mathieu functions of even order [34, 35, 23], the ground-state wavefunction being

$$\psi_{\xi,\langle n\rangle}(\phi) = \frac{1}{\sqrt{\pi}} ce_0\left(\frac{\phi}{2}, -\frac{2(1-\xi)}{\xi}\right) e^{i\langle n\rangle\phi}. \tag{46}$$

The MUS uncertainties are then

$$\Delta n_{(\xi)} = \left(\frac{1}{\pi}\int_0^{2\pi}\left[\frac{\mathrm{d}}{\mathrm{d}\phi}ce_0\left(\frac{\phi}{2},\theta\right)\right]^2 \mathrm{d}\phi\right)^{1/2}$$
$$\Delta\phi_{(\xi)} = \arccos\left(\frac{1}{\pi}\int_0^{2\pi}\left[ce_0\left(\frac{\phi}{2},\theta\right)\right]^2 \cos\phi\,\mathrm{d}\phi\right) \tag{47}$$

where $\theta = -2(1-\xi)/\xi$. This relation between the Δn and $\Delta\phi$ uncertainties is a limiting one for the harmonic oscillator—no state can reach these values, but it is possible to get arbitrarily close values by increasing $\langle n\rangle$ sufficiently. This relation is depicted graphically in figure 7 (broken curve)—this curve is the same as the limiting one for the Pegg–Barnett model. As may be checked with increasing N and by expanding (30) into a Taylor series, for states with zero $\bar{n}$ and sufficiently small Δn the essential matrix elements of the uncertainty Hamiltonian (30) can be made arbitrarily close to those of (44); therefore the limiting relation of the finite-dimensional Hilbert space models ($\Delta n \ll N$) is the same as for the quantum rotator and highly excited harmonic oscillator ($\Delta n \ll \langle n\rangle$).

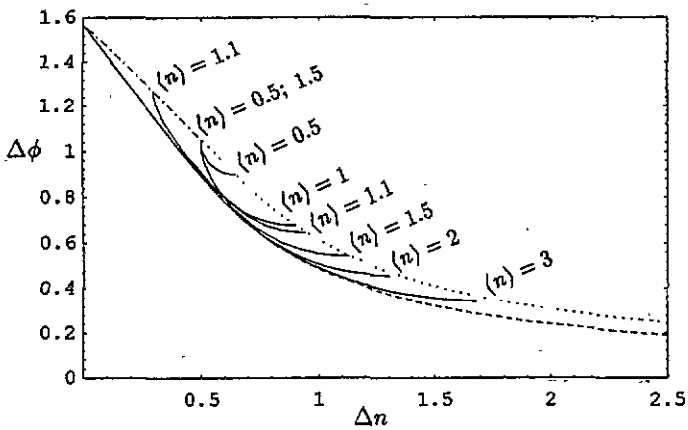

Figure 7. Relation between the Δn and $\Delta\phi$ MUS uncertainties under the condition of a given mean photon number. (– – –): the limit of $\langle n\rangle \to \infty$. (— · —): $\Delta\phi = \arccos\Delta n$: beginning of curves with non-integer $\langle n\rangle$. (······): $\Delta n \sim \Delta\phi$ dependence for the phase optimized states, end of the uncertainty curves. (——): $\Delta n \sim \Delta\phi$ uncertainties for various given $\langle n\rangle$.

We can get a very good approximation of the ground state of (44) using the Ritz variational method. As the test function we can take the von Mises distribution

$$\psi_A(\phi) = (2\pi I_0(A))^{-1/2}\exp\left(\frac{A}{2}\cos\phi + \mathrm{i}\langle n\rangle\phi\right) \tag{48}$$

in the phase representation, which is

$$\langle n|\psi_A\rangle = \frac{I_{n-\langle n\rangle}(A/2)}{\sqrt{I_0(A)}} \tag{49}$$

in the n representation. Here I_n stands for the modified Bessel function. Let us mention that these states minimize the uncertainty product $\Delta L_z\Delta\sin\phi/\langle\cos\phi\rangle$ [2, 22]; their relationship to the harmonic oscillator was discussed in [24]. The $\Delta n \sim \Delta\phi$ relation for these states is given by the parametric function

$$\Delta n_{(A)} = \frac{\sqrt{\sum_n I_n^2(A/2)n^2}}{I_0(A)} \qquad \Delta\phi_{(A)} = \arccos\frac{I_1(A)}{I_0(A)}. \tag{50}$$

The difference between this approximate $\Delta\phi$ and the actual limiting value is very small, at most about 0.9%.

4.3. Phase uncertainty versus number uncertainty; arbitrary mean photon number

Let us now turn our attention to the general problem of finding states which minimize the photon number uncertainty and the phase uncertainty with a given $\langle n\rangle$. Lukš *et al* [23] got very close to the solution of this problem (for special values of $\langle n\rangle$) starting from the minimizing states of the plane rotator (46) and omitting the Fourier coefficients with negative indices. This approximation is very good due to fast convergence of the Mathieu function Fourier expansion. Here we will consider the MUS states for arbitrary $\langle n\rangle$ and with arbitrary precision. Such states can be found as the ground states of a two-parameter uncertainty Hamiltonian

$$\hat{H}(\xi,\mu) = \xi\hat{n}^2 + \mu\hat{n} - (1-\xi)\hat{C}\,. \tag{51}$$

Here the parameter ξ takes values from the interval $(0, 1]$, whereas μ can take arbitrary real values, even negative ones. We can easily check that such ground states are the MUS: let the mean photon number of the ground state be $\langle n\rangle$; then no state with the same mean photon number and the same $\langle\hat{C}\rangle$ can have smaller $\langle\hat{n}^2\rangle$ and thus smaller Δn, and similarly no states with the same $\langle n\rangle$ and the same $\langle\hat{n}^2\rangle$ can have larger $\langle\hat{C}\rangle$ and thus smaller $\Delta\phi$.

The resulting MUS relations for $\Delta\phi$ and Δn are depicted in figure 7 and can be described as follows. If the mean photon number is integer, then the MUS curve begins at $\Delta n = 0$ and $\Delta\phi = \pi/2$, i.e. the beginning corresponds to a Fock state $|n\rangle$, $n = \langle n\rangle$. If $\langle n\rangle$ is not an integer, say $\langle n\rangle = [[\langle n\rangle]] + p$ (where $[[x]]$ is the largest integer not exceeding x), then the state with minimum possible Δn uncertainty is the superposition of two neighbouring Fock states $\sqrt{1-p}|[[\langle n\rangle]]\rangle + \sqrt{p}|[[\langle n\rangle]]+1\rangle$. The n uncertainty is $\Delta n = \sqrt{p(1-p)}$ and the phase uncertainty is $\Delta\phi = \arccos\sqrt{p(1-p)}$. Therefore, all MUS curves for states with non-integer $\langle n\rangle$ begin at points of the curve $\Delta\phi = \arccos(\Delta n)$ for $\Delta n \in (0, \frac{1}{2}]$. Every MUS curve then ends at the curve representing uncertainties Δn and $\Delta\phi$ for the phase optimized states with given $\langle n\rangle$. Then no increasing of Δn can decrease the phase uncertainty. Note also that for $\langle n\rangle$ sufficiently large (and integer) MUS curves approach the quantum rotator limiting curve of uncertainties (47) as they should.

5. Conclusion

In this paper we have discussed a possible way for finding MUSs for a relatively wide class of different uncertainties. The method is based on solving ground-state problems of some 'uncertainty Hamiltonians'. It is very effective, especially when we solve this problem numerically and can use software routines for finding eigenvalues of matrices. When trying to perform analytical calculations, we usually obtain the same equations like when using the method of the Lagrange multipliers. Nevertheless, the main idea of the method of uncertainty Hamiltonians encourages us to take advantage of the all known mathematical apparatus used for approximately solving the stationary Schrödinger equation.

Our main aim here was to find the limiting relations between the photon number uncertainty and phase uncertainty (the definition of phase uncertainty being based on the Bandilla–Paul dispersion). We have approached this point in two ways: (i) working in finite-dimensional Hilbert spaces and then increasing the dimension to infinity (the Pegg–Barnett model) and (ii) considering the uncertainty relation with a fixed mean photon number and then increasing this quantity.

The results can be summarized as follows. Given a dimension of the Hilbert space N and the uncertainty Δn, we can find the limiting uncertainty $\Delta\phi$ (which means that no state can have smaller $\Delta\phi$). Similarly, if we consider a usual harmonic oscillator, given a mean photon number $\langle n\rangle$ and the uncertainty Δn, we can find the limiting value for the $\Delta\phi$ uncertainty. These uncertainty relations cannot (except for a few of the simplest cases) be expressed as some elementary functions, but the values can be calculated with arbitrary precision. The $\Delta\phi$ uncertainties calculated in the two ways approach each other in the limit $N \to \infty$ and $\langle n\rangle \to \infty$.

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