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Quantum state measurement of multimode light pulses[1]

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A method for measuring internal quantum correlations and multimode density matrices of optical pulses is proposed. In balanced homodyne detection a signal pulse and sequences of short local-oscillator pulses are superimposed and the time-integrated difference-count statistics is recoreded. For chosen distances between the test pulses, the phases and relative intensities of the pulses are varied from measurement to measurement. Using a sequence of N test pulses, the quantum statistics of the signal pulse can then be obtained in terms of N correlated non-monochromatic modes. In particular, the determination of the N-mode density matrix in a field-strength basis can be accomplished with $(N+1)$ Fourier integrals. The method also applies to the measurement of the quantum state of a correlated N-mode field whose modes are separable. In this case pre-superimposed signal-field modes must be combined with a local-oscillator mode.

1. Introduction

One of the most powerful methods for measuring the quantum states of optical fields has been balanced homodyne detection. It is well known that when a signal mode and a strong local-oscallator (LO) mode are mixed by means of a 50%:50% beam splitter and the difference photocurrent of two photodetectors in the output channels of the beam splitter is measured, the field-strength statistics of the signal mode for a certain phase

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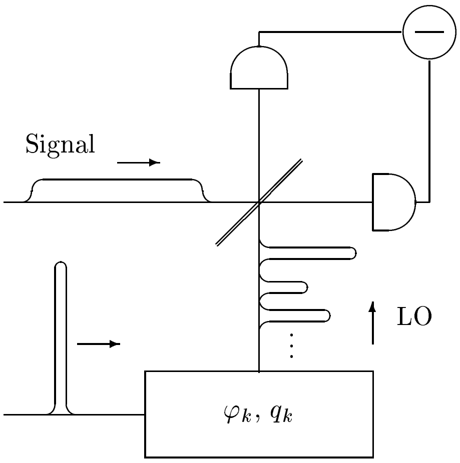
parameter can directly be obtained, see, e.g., [1]. As was shown theoretically [2] and demonstrated experimentally [3], from the field-strength distributions for all phases in a π interval the quantum state of the signal mode can be obtained. In the experiments in [3] the Wigner function was reconstructed tomographically and used to calculate the density matrix in a field-strength basis by performing a Fourier transformation. Later it was shown that the quantum state can also be obtained by direct sampling of the density matrix in various bases from the measured data [4,5]. The method can also be extended to the measurement of the quantum state of multimode optical fields whose modes can be used separately as input fields in multiport homodyning [6,7]

Recently experiments have been performed in which a signal pulse and a LO pulse that is short compared with the signal pulse are combined in order to measure the photon-number statistics of the signal pulse at different times in the pulse [8]. Using a train of well-separated short LO pulses and varying the phases and relative intensities, we show that the correlated multimode quantum state of the signal pulse can be obtained by direct sampling of multimode density matrices from the measured data [10]. The method can easily be modified in order to measure also density matrices of correlated multimode fields whose modes are separated from each other.

2. Measurement scheme

To measure the internal quantum statistics of optical pulses, an apparatus is desired that performs an appropriate mode decomposition of the pulses without introduction of additional noise and measures the quantum statistics of the correlated modes. This can be achieved by a homodyne detection scheme shown in the figure. The signal pulse



Scheme of measurement [10]. A signal pulse and a train of strong LO pulses that are short compared with the signal pulse are combined by a 50%:50% beam splitter, and the time-integrated difference-count statistics in the output channels is measured. The train of LO pulses is produced interferometrically, so that the pulse distances, relative phases, and intensities can be controlled.

consists, in general, of a continuum of monochromatic modes with photon destruction and creation operators $\hat{b}(\omega)$ and $\hat{b}^\dagger(\omega')$, respectively, $[\hat{b}(\omega),\, \hat{b}^\dagger(\omega')] = \delta(\omega' - \omega)$. Let us suppose that the LO pulses are prepared in coherent states, so that the positive-frequency part of the kth pulse centered at time t_k can be described by a function

$\gamma_k g_k(t)$, where $g_k(t)$ is assumed to be normalized to unity and γ_k is a complex number. We now consider the operators

$$\hat{a}_k = \int d\omega\, \tilde{g}_k^*(\omega)\, \hat{b}(\omega), \tag{1}$$

where $\tilde{g}_k(\omega)$ is the Fourier transform of $g_k(t)$, $\tilde{g}_k(\omega) \equiv (2\pi)^{-1/2} \int d\omega\, g_k(t) \exp(i\omega t)$. It can be shown that when the LO pulses $g_k(t)$ and $g_{k'}(t)$, $k \neq k'$, are (approximately) nonoverlapping, then the operators $\hat{a}_k$ and $\hat{a}_{k'}^\dagger$ satisfy the standard bosonic commutation relation $[\hat{a}_k, \hat{a}_{k'}^\dagger] = \delta_{kk'}$. Hence, the train of N LO pulses can be used to introduce a set of N nonmonochromatic modes [9] and "probe" the correlation statistics of the signal pulse in terms of these modes. In particular, measurement of the integrated difference-count statistics can be shown to be equivalent to measurement of the sum of signal-pulse field strengths

$$\hat{F} = \sum_{k=1}^{N} q_k \hat{F}_k(\varphi_k). \tag{2}$$

Here,

$$\hat{F}_k(\varphi_k) = 2^{-1/2} \left(\hat{a}_k e^{-i\varphi_k} + \hat{a}_k^\dagger e^{i\varphi_k} \right) \tag{3}$$

is the field strength associated with the kth pulse-like nonmonochromatic mode centered at time t_k in the signal pulse and q_k is a non-negative real parameter given by the (relative) amplitude of the kth LO pulse. Measuring the distribution of the sum field strength (3) for all values of the (relative) amplitudes q_k ($q_k \geq 0$) and phases φ_k ($\phi_k \in$ π intervals), we can obtain the quantum statistics of the signal pulse within the frame of an N-mode density matrix.

3. Intermodal correlations

For the sake of transparency let us consider the simplest case when the signal pulse and only two short nonoverlapping LO pulses centered at the times t_1 and t_2 in the signal pulse are superimposed. In this case, measurement of the time-integrated difference-count statistics yields the statistics of the sum of the field strengths of two modes of the signal pulse,

$$\hat{F} = \hat{F}(\varphi, \Delta\varphi, q) = \hat{F}_1(\varphi) + q\,\hat{F}_2(\varphi + \Delta\varphi) \tag{4}$$

(i.e., $q_1 = 1$, $q_2 = q$). The measured moments of $\hat{F}$ are related to the moments and correlation functions of the signal pulse at the two times t_1 and t_2 in the pulse as

$$\langle \hat{F}^n \rangle = \sum_{m=0}^{n} \binom{n}{m} q^m \langle \hat{F}_1^{n-m}(\varphi) \hat{F}_2^m(\varphi + \Delta\varphi) \rangle. \tag{5}$$

Measuring $\langle \hat{F}^n \rangle$ for $(n+1)$ values of q, from Eq. (5) we obtain a set of $(n+1)$ linear algebraic equations whose solution yields the signal-pulse moments and correlation functions $\langle \hat{F}_1^{n-m}(\varphi) \hat{F}_2^m(\varphi + \Delta\varphi) \rangle$ for the chosen values of φ and $\Delta\varphi$. Varying φ and $\Delta\varphi$, the procedure can be repeated many times to obtain the φ and $\Delta\varphi$ dependences of the moments and correlation functions. In the limit when the number of measurements goes to infinity all the moments and correlations can be obtained, the knowledge of which is equivalent to knowledge of the two-mode density matrix.

If we assume that the signal and the LO come from different light sources, then the phase φ cannot be controlled and the phase-averaged (even) moments

$$\overline{\langle \hat{F}^n \rangle} \equiv \frac{1}{2\pi} \int d\varphi \, \langle \hat{F}^n \rangle \tag{6}$$

can only be measured. Eq.(5) then modifies to

$$\overline{\langle \hat{F}^n \rangle} = \sum_{m=0}^{n} \binom{n}{m} q^m \overline{\langle \hat{F}_1^{n-m}(\varphi) \hat{F}_2^m(\varphi + \Delta\varphi) \rangle}, \tag{7}$$

n being even. For example, when $n=2$ Eq. (7) enables us to calculate the quantities $\overline{\langle \hat{F}_k^2(\varphi) \rangle}$ and $\overline{\langle \hat{F}_1(\varphi) \hat{F}_2(\varphi + \Delta\varphi) \rangle}$, $k=1,2$. The moments $\overline{\langle \hat{F}_k^2(\varphi) \rangle}$ are closely related to the mean numbers of photons $\langle \hat{n}_1 \rangle$ and $\langle \hat{n}_2 \rangle$ at the two times t_1 and t_2, respectively, in the signal pulse. The cross term

$$\overline{\langle \hat{F}_1(\varphi) \hat{F}_2(\varphi + \Delta\varphi) \rangle} = \tfrac{1}{2} \langle \hat{a}_1^\dagger \hat{a}_2 \rangle e^{-i\Delta\varphi} + \text{c.c.} \tag{8}$$

can be regarded as the second-order coherence function that probes the effect of signal-pulse interference at a "temporal double slit". It should be noted that the photon-number correlation $\langle \hat{n}_1 \hat{n}_2 \rangle$, which for stationary ergodic fields is usually measured in a Hanbury-Brown and Twiss experiment, is among the terms available in the case when $n=4$.

4. Multimode density matrices

We now turn to the problem of determining the density matrix. For simplicity let us again restrict attention to the two-mode case. We first introduce the single-mode field-strength eigenvectors $|\mathcal{F}_k, \varphi_k\rangle$ satisfying the eigenvalue equation $\hat{F}_k(\varphi_k)\,|\mathcal{F}_k, \varphi_k\rangle = \mathcal{F}_k\,|\mathcal{F}_k, \varphi_k\rangle$, where $\hat{F}_k(\varphi_k)$ is given by Eq. (3), $k=1,2$. Accordingly, $|\mathcal{F}_1, \mathcal{F}_2, \varphi_1, \varphi_2\rangle = |\mathcal{F}_1, \varphi_1\rangle |\mathcal{F}_2, \varphi_2\rangle$ are the eigenvectors of $\hat{F} = \hat{F}_1(\varphi_1) + \hat{F}_2(\varphi_2)$ belonging to the eigenvalues $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$. The two-mode joint field-strength distribution of the signal pulse is then given by $p_{\mathrm{j}}(\mathcal{F}_1, \mathcal{F}_2, \varphi_1, \varphi_2) = \langle \mathcal{F}_1, \mathcal{F}_2, \varphi_1, \varphi_2|\,\hat{\varrho}\,|\mathcal{F}_1, \mathcal{F}_2, \varphi_1, \varphi_2\rangle$, where $\hat{\varrho}$ is the two-mode (reduced) density operator of the pulse. The probability distribution of the measured sum-field strength (2) can be given by

$$p_{\mathrm{s}}(\mathcal{F}, \alpha, \psi_1, \psi_2) = \int \mathrm{d}\mathcal{F}_1 \int \mathrm{d}\mathcal{F}_2 \, p_{\mathrm{j}}(\mathcal{F}_1, \mathcal{F}_2, \psi_1, \psi_2)\, \delta(\mathcal{F} - \mathcal{F}_1 \cos\alpha - \mathcal{F}_2 \sin\alpha), \tag{9}$$

where we have substituded in Eq. (2) for q_1 and q_2, respectively, $\cos\alpha$ and $\sin\alpha$ ($\alpha \in (0, \pi/2)$). To obtain the two-mode density matrix, we note that in a field-strength basis it is simply given by [4,6]

$$\begin{aligned} &\langle \mathcal{F}_1 - \mathcal{F}_1', \mathcal{F}_2 - \mathcal{F}_2', \varphi_1, \varphi_2|\,\hat{\varrho}\,|\mathcal{F}_1 + \mathcal{F}_1', \mathcal{F}_2 + \mathcal{F}_2', \varphi_1, \varphi_2\rangle \\ &\qquad = \left(\frac{1}{2\pi}\right)^2 \int \mathrm{d}y_1 \int \mathrm{d}y_2 \, e^{-i(y_1 \mathcal{F}_1 + y_2 \mathcal{F}_2)}\, \Psi(z_1, z_2, \psi_1, \psi_2), \end{aligned} \tag{10}$$

where

$$z_k \equiv z_k(y_k, \mathcal{F}'_k) = \sqrt{{y_k}^2 + (2\mathcal{F}'_k)^2}, \tag{11}$$

$$\psi_k \equiv \psi_k(y_k, \mathcal{F}'_k) = \varphi_k - \mathrm{arccot}[y_k/(2\mathcal{F}'_k)] \tag{12}$$

($\mathcal{F}'_k \geq 0$), and $\Psi(z_1, z_2, \psi_1, \psi_2)$ is the characteristic function of the joint probability distribution,

$$\Psi(z_1, z_2, \psi_1, \psi_2) = \int \mathrm{d}\mathcal{F}_1 \int \mathrm{d}\mathcal{F}_2\, e^{i(z_1\mathcal{F}_1 + z_2\mathcal{F}_2)} p_{\mathrm{j}}(\mathcal{F}_1, \mathcal{F}_2, \psi_1, \psi_2). \tag{13}$$

The key point here is to take advantage of the connection between the probability distribution of the sum, Eq. (9), and the characteristic function of the joint probability distribution, Eq. (13), which can easily be derived to be

$$\Psi(z_1, z_2, \psi_1, \psi_2) = \int \mathrm{d}\mathcal{F}\, e^{iz\mathcal{F}}\, p_{\mathrm{s}}(\mathcal{F}, \alpha, \psi_1, \psi_2), \tag{14}$$

where

$$z \equiv z(z_1, z_2) = \sqrt{z_1^2 + z_2^2}, \tag{15}$$

$$\alpha \equiv \alpha(z_1, z_2) = \arctan(z_2/z_1). \tag{16}$$

Thus, combining Eqs. (10) and (14) we arrive at the reconstruction formula

$$\begin{aligned} &\langle \mathcal{F}_1 - \mathcal{F}'_1, \mathcal{F}_2 - \mathcal{F}'_2, \varphi_1, \varphi_2 |\, \hat{\varrho}\, | \mathcal{F}_1 + \mathcal{F}'_1, \mathcal{F}_2 + \mathcal{F}'_2, \varphi_1, \varphi_2 \rangle \\ &\quad = \left(\frac{1}{2\pi}\right)^2 \int \mathrm{d}y_1 \int \mathrm{d}y_2\, e^{-i(y_1\mathcal{F}_1 + y_2\mathcal{F}_2)} \int \mathrm{d}\mathcal{F}\, e^{iy\mathcal{F}}\, p_{\mathrm{s}}(\mathcal{F}, \beta, \psi_1, \psi_2), \end{aligned} \tag{17}$$

where [according to Eqs. (11), (15), and (16)]

$$y \equiv y(y_1, y_2, \mathcal{F}'_1, \mathcal{F}'_2) = \sqrt{y_1^2 + y_2^2 + (2\mathcal{F}'_1)^2 + (2\mathcal{F}'_2)^2}\,, \tag{18}$$

$$\beta \equiv \beta(y_1, y_2, \mathcal{F}'_1, \mathcal{F}'_2) = \arctan\sqrt{\frac{y_2^2 + (2\mathcal{F}'_2)^2}{y_1^2 + (2\mathcal{F}'_1)^2}}\,, \tag{19}$$

and ψ_k is given in Eq. (12).

5. Conclusions

Equation (17) reveals that the two-mode density matrix can be obtained from the sum-field-strength distributions by a threefold Fourier transformation. The equation can be used for direct sampling of the density matrix from the measured data quite similar to the algorithm in [4]. Using the sampling function given in [5], from Eq. (17) the density matrix in the photon-number basis can be obtained as well. In the paper we have considered perfect detection. Extension of the results to nonperfect detection requires inclusion in the theory of the effect of the additional noise associated with the losses (see, for details, [10]).

It should be pointed out that the method is rather universal and not restricted to pulses. In particular, measurements of the various intermodal moments can be regarded as coherence measurements which of course apply to arbitrary optical fields. Compared with the usual coherence measurements, they only contain ensemble averaging. Since there is no temporal averaging, there is no necessity of restriction to stationary fields. Moreover, the method can also be used to measure intermodal moments and density matrices of spatially separated modes. In this case the modes can interferometrically be combined in order to obtain a pre-superimposed signal mode who is detected in four-port balanced homodyning (see, for details, [10]).

We have restricted attention to the case of detecting two modes. A generalization of the results to the case when N modes are detected is straightforward. In particular, the determination of the N-mode density matrix can be accomplished with $(N+1)$ Fourier integrals – one integral for obtaining the characteristic function of the density matrix from the measured data and one integral per mode.

After preparing this paper we received a preprint with similar problems: M.G. Raymer, D.F. McAlister, and U. Leonhardt, Two-mode quantum-optical state measurement: Sampling the joint density matrix (submitted to *Phys. Rev. A*).

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